DIFFERENTIAL AND INTEGRAL INEQUALITIES

Volume 55-I

V. Lakshmikantham &
S. Leela

DIFFERENTIAL AND INTEGRAL INEQUALITIES

Theory and Applications

Volume I

ORDINARY DIFFERENTIAL EQUATIONS

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Volume I

ORDINARY DIFFERENTIAL EQUATIONS

V. LAKSHMIKANTHAM and S. LEELA

University of Rhode Island
Kingston, Rhode Island

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Preface

This volume constitutes the first part of a monograph on theory and applications of differential and integral inequalities. The entire work, as a whole, is intended to be a research monograph, a guide to the literature, and a textbook for advanced courses. The unifying theme of this treatment is a systematic development of the theory and applications of differential inequalities as well as Volterra integral inequalities. The main tools for applications are the norm and the Lyapunov functions. Familiarity with real and complex analysis, elements of general topology and functional analysis, and differential and integral equations is assumed.

The theory of differential inequalities depends on integration of differential inequalities or what may be called the general comparison principle. The treatment of this theory is not for its own sake. The essential unity is achieved by the wealth of its applications to various qualitative problems of a variety of differential systems.

The material of the present volume is divided into two sections. The first section consisting of four chapters deals with ordinary differential equations while the second section is devoted to Volterra integral equations. The remaining portion of the monograph, which will appear as a second volume, is concerned with differential equations with time lag, partial differential equations of first order, parabolic and hyperbolic respectively, differential equations in abstract spaces including nonlinear evolution equations and complex differential equations types.

The vector notation and vectorial inequalities are used freely throughout the book. Also, because of the several allied fields covered, it becomes convenient to use the same letter with different meanings in different situations. This, however, should not cause confusion, since it is spelled out wherever necessary.

The notes at the end of each chapter indicate the sources which have

been consulted and those whose ideas are developed. Some sources which are closely related but not included in the book are also given for guidance.

We wish to express our warmest thanks to our colleague Professor C. Corduneanu for reading the manuscript and suggesting improvements. Our thanks are also due to Professors J. Hale, N. Onuchic, and C. Olech for their helpful suggestions.

We are immensely pleased that our monograph appears in a series inspired and edited by Professor R. Bellman and we wish to express our gratitude and warmest thanks for his interest in this book.

V. LAKSHMIKANTHAM
S. LEELA

Kingston, Rhode Island
December, 1968

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Chapter 1

1.0. Introduction

This chapter is an introduction to the theory of differential inequalities and therefore forms a basis of the remaining chapters. After sketching the preliminary existence and continuation of solutions of an initial value problem for ordinary differential equations, we develop fundamental results involving differential inequalities. Basic comparison theorems that form the core of the monograph are treated in detail. While considering the system of differential inequalities (finite or infinite), we find it convenient to utilize the notion minimax solutions, and consequently our treatment rests on this notion. Certain useful integral inequalities that can be reduced to the theory of differential inequalities are also presented. Some results on differential inequalities of Caratheodory's type are also dealt with.

1.1. Existence and continuation of solutions

Let R^n denote the real n-dimensional, euclidean space of elements $u = (u_1, u_2, \dots, u_n)$. Sometimes, we shall denote also the $(n + 1)$-tuple $(t, u_1, u_2, \dots, u_n)$ as an element, and R^{n+1} shall denote the space of elements $(t, u_1, u_2, \dots, u_n)$ or (t, u). Let $\| u \|$ be any convenient norm. As usual, we shall use R instead of R^1. Let E be an open (t, u)-set in R^{n+1}. We shall mean by $C[E, R^n]$ the class of continuous mappings from E into R^n. If f is a member of this class, one writes $f \in C[E, R^n]$. Let us consider a system of first-order differential equations with an initial condition

$$u' = g(t, u), \qquad u(t_0) = u_0, \tag{1.1.1}$$

where $u' = du/dt$, $u_0 = (u_{10}, u_{20}, \dots, u_{n0})$, and $g \in C[E, R^n]$. A solution of the initial value problem (1.1.1) is a differentiable function of t such

that $u(t_0) = u_0$, $(t, u(t)) \in E$, and $u'(t) = g(t, u(t))$ for a t-interval J containing t_0. This means that $u(t)$ has a continuous derivative. From these requirements on the continuous function $u(t)$, it follows that it satisfies the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} g(s, u(s))\, ds, \qquad t \in J.$$

In order to prove the classical Peano's existence theorem, we have to introduce the notion of an equicontinuous family of functions.

DEFINITION 1.1.1. A family of functions $F = \{f(u)\}$ defined on some u-set $E \subset R^n$ is said to be equicontinuous if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$, independent of $f \in F$ and also $u_1, u_2 \in E$, such that $\|f(u_1) - f(u_2)\| < \epsilon$ whenever $\| u_1 - u_2 \| < \delta$.

The following theorem shows the fundamental property of such a family of functions, the proof of which will be omitted.

THEOREM 1.1.1. (Ascoli-Arzela). Let $F = \{f\}$ be a sequence of functions defined on a compact u-set $E \subset R^n$, which is equicontinuous and equibounded. Then, there exists a subsequence $\{f_n\}$, $n = 1, 2,...$, which is uniformly convergent on E.

THEOREM 1.1.2. (Peano's Existence Theorem). Let $g \in C[R_0, R^n]$, where R_0 is the set $[(t, u)\colon t_0 \leqslant t \leqslant t_0 + a, \| u - u_0 \| \leqslant b]$; $\| g(t, u)\| \leqslant M$ on R_0. Then, the initial value problem (1.1.1) possesses at least one solution $u(t)$ on $t_0 \leqslant t \leqslant t_0 + \alpha$, where $\alpha = \min(a, b/M)$.

Proof. Let $u_0(t)$ be a continuously differentiable function, on $[t_0 - \delta, t_0]$, $\delta > 0$, such that $u_0(t_0) = u_0$, $\| u_0(t) - u_0 \| \leqslant b$, and $\| u_0'(t) \| \leqslant M$. For $0 < \epsilon \leqslant \delta$, we define a function $u_\epsilon(t) = u_0(t)$ on $[t_0 - \delta, t_0]$ and

$$u_\epsilon(t) = u_0 + \int_{t_0}^{t} g(s, u_\epsilon(s - \epsilon))\, ds \tag{1.1.2}$$

on $[t_0, t_0 + \alpha_1]$, where $\alpha_1 = \min(\alpha, \epsilon)$. Observe that $u_\epsilon(t)$ is differentiable and

$$\| u_\epsilon(t) - u_0 \| \leqslant b \tag{1.1.3}$$

on $[t_0 - \delta, t_0 + \alpha_1]$. If $\alpha_1 < \alpha$, we can use (1.1.2) to extend $u_\epsilon(t)$ as a continuously differentiable function over $[t_0 - \delta, t_0 + \alpha_2]$, $\alpha_2 = \min(\alpha, 2\epsilon)$, such that (1.1.3) holds. Continuing in this way, $u_\epsilon(t)$ can be defined over $[t_0 - \delta, t_0 + \alpha]$ so that it has a continuous derivative and

satisfies (1.1.3) on the same interval. Furthermore, $\| u_\epsilon'(t)\| < M$, and therefore $\{u_\epsilon(t)\}$ forms a family of equicontinuous and uniformly bounded functions. An application of Theorem 1.1.1 shows the existence of a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > \cdots \epsilon_n \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly on $[t_0 - \delta, t_0 + \alpha]$. Since g is uniformly continuous, we obtain that $g(t, u_{\epsilon_n}(t - \epsilon_n))$ tends uniformly to $g(t, u(t))$ as $n \to \infty$, and, hence, term-by-term integration of (1.1.2) with $\epsilon = \epsilon_n$, $\alpha_1 = \alpha$ yields

$$u(t) = u_0 + \int_{t_0}^{t} g(s, u(s))\, ds.$$

This proves that $u(t)$ is a solution of (1.1.1).

The following corollary of Peano's Theorem is useful in applications.

COROLLARY 1.1.1. Let E be an open (t, u)-set in R^{n+1} and E_0 be a compact subset of E. Suppose that $g \in C[E, R^n]$ and $\| g(t, u)\| \leqslant M$ on E. Then, there exists an $\alpha = \alpha(E, E_0, M)$ such that, if $(t_0, u_0) \in E_0$, (1.1.1.) has a solution, and every solution exists on $[t_0, t_0 + \alpha]$.

In that case, when g is not bounded on E, we can replace the set E by an open subset E_1 having a compact closure in E and containing E_0.

The next theorem deals with the problem of extending the solutions up to the boundary of E.

THEOREM 1.1.3. Let E be an open (t, u)-set in R^{n+1}, and let $g \in C[E, R^n]$ and $u(t)$ be a solution of (1.1.1) on some interval $t_0 \leqslant t \leqslant a_0$. Then $u(t)$ can be extended as a solution to the boundary of E.

Proof. Let $E_1, E_2, \ldots$ be open subsets of E such that $E = \bigcup E_n$; the closures $\bar{E}_1, \bar{E}_2, \ldots$ are compact, and $\bar{E}_n \subset E_{n+1}$. It then follows from Corollary 1.1.1 that there exists an $\epsilon_n > 0$ such that, if $(t_0, u_0) \in \bar{E}_n$, all solutions of (1.1.1) exist on $t_0 \leqslant t \leqslant t_0 + \epsilon_n$.

Choose n_1 so large that $(a_0, u(a_0)) \in \bar{E}_{n_1}$. Then, $u(t)$ can be extended over an interval $[a_0, a_0 + \epsilon_{n_1}]$, and, if $(a_0 + \epsilon_{n_1}, u(a_0 + \epsilon_{n_1})) \in \bar{E}_{n_1}$, $u(t)$ can be further extended over $[a_0 + \epsilon_{n_1}, a_0 + 2\epsilon_{n_1}]$. This argument can be repeated until we get the extension of $u(t)$ over the interval $t_0 \leqslant t \leqslant a_1$, where $a_1 = a_0 + N_1\epsilon_{n_1}$, N_1 is an integer $\geqslant 1$, such that $(a_1, u(a_1)) \notin \bar{E}_{n_1}$.

Choose n_2 so large that $(a_1, u(a_1)) \in \bar{E}_{n_2}$. Arguing as before, we arrive at an integer $N_2 \geqslant 1$ such that $u(t)$ can be extended over $t_0 \leqslant t \leqslant a_2$, $a_2 = a_1 + N_2\epsilon_{n_2}$, and $(a_2, u(a_2)) \notin \bar{E}_{n_2}$.

Proceeding in this way, we are led to a sequence of integers

$n_1 < n_2 < \cdots$ and numbers $a_0 < a_1 < a_2 < \cdots$ such that $u(t)$ has an extension over $[t_0, a)$, where $a = \lim_{k\to\infty} a_k$ and that $(a_k, u(a_k)) \notin \bar{E}_{n_k}$. Thus, the sequence $\{a_k, u(a_k)\}$ is either unbounded or has a cluster point on the boundary of E.

To show that $u(t)$ tends to the boundary of E as $t \to a$, we must show that no limit point of $\{t_k, u(t_k)\}$ is an interior point of E as $t_k \to a$. Since this follows from the lemma below, the theorem is proved.

LEMMA 1.1.1. Let $g \in C[E, R^n]$, where E is an open (t, u)-set in R^{n+1}. Let $u(t)$ be a solution of (1.1.1) on an interval $t_0 \leqslant t < a$, $a < \infty$. Assume that there exists a sequence $\{t_k\}$ such that $t_0 \leqslant t_k \to \infty$ as $k \to \infty$ and $u^0 = \lim_{k\to\infty} u(t_k)$ exists. If $g(t, u)$ is bounded on the intersection of E and a neighborhood of (a, u^0), then

$$\lim_{t\to a} u(t) = u^0. \tag{1.1.4}$$

If, in addition, $g(a, u^0)$ is defined such that $g(t, u)$ is continuous at (a, u^0), then $u(t)$ is continuously differentiable on $[t_0, a]$ and is a solution of (1.1.1) on $[t_0, a]$.

Proof. Let $\epsilon > 0$ be sufficiently small. Consider the set $\mathring{R}$: $0 \leqslant a - t \leqslant \epsilon$, $\| u - u^0 \| \leqslant \epsilon$. Let $M(\epsilon)$ be so large that $\| g(t, u)\| \leqslant M(\epsilon)$ for $(t, u) \in E \cap \mathring{R}$. If, for k sufficiently large, $0 < a - t_k \leqslant \epsilon/2M(\epsilon)$ and $\| u(t_k) - u^0 \| \leqslant \epsilon/2$, then

$$\| u(t) - u(t_k)\| < M(\epsilon)(a - t_k) \leqslant \epsilon/2 \tag{1.1.5}$$

for $t_k \leqslant t < a$. If this is not true, there is a t_1 such that $t_k < t_1 < a$, $\| u(t_1) - u(t_k) \| = M(\epsilon)(a - t_k) \leqslant \epsilon/2$. It therefore follows that

$$\| u(t) - u^0 \| \leqslant \tfrac{1}{2}\epsilon + \| u(t_k) - u^0 \| \leqslant \epsilon \qquad \text{for} \quad t_k \leqslant t < t_1 .$$

This implies $\| u'(t)\| \leqslant M(\epsilon)$ for $t_k \leqslant t \leqslant t_1$. Consequently,

$$\| u(t_1) - u(t_k)\| \leqslant M(\epsilon)(t_1 - t_k) < M(\epsilon)(a - t_k).$$

This proves (1.1.5), which, in turn, shows that (1.1.4) holds. The last part of the lemma follows from the fact that

$$u'(t) = g(t, u(t)) \to g(a, u^0) \qquad \text{as} \quad t \to a.$$

COROLLARY 1.1.2. Let $g \in C[E, R^n]$, where

$$E = [(t, u): t_0 \leqslant t \leqslant t_0 + a\ (a < \infty), u \in R^n].$$

Let $u(t)$ be a solution of (1.1.1). Then the largest interval of existence of $u(t)$ is either $[t_0, t_0 + a]$ or $[t_0, \delta)$, $\delta \leqslant t_0 + a$ and $\| u(t) \| \to \infty$ as $t \to \delta$.

1.2. Scalar differential inequalities

We adopt the following notation for Dini derivatives:

$$D^+u(t) = \limsup_{h \to 0^+} h^{-1}[u(t+h) - u(t)],$$

$$D_+u(t) = \liminf_{h \to 0^+} h^{-1}[u(t+h) - u(t)],$$

$$D^-u(t) = \limsup_{h \to 0^-} h^{-1}[u(t+h) - u(t)],$$

$$D_-u(t) = \liminf_{h \to 0^-} h^{-1}[u(t+h) - u(t)],$$

where $u \in C[(t_0, t_0 + a), R]$. When $D^+u(t) = D_+u(t)$, the right derivative will be denoted by $u'_+(t)$. Similarly, $u'_-(t)$ denotes the left derivative.

DEFINITION 1.2.1. Let E be an open (t, u)-set in R^2 and $g \in C[E, R]$. Consider the scalar differential equation with an initial condition

$$u' = g(t, u), \qquad u(t_0) = u_0. \tag{1.2.1}$$

Suppose $v \in C[[t_0, t_0 + a), R]$, $v'_+(t)$ exists for $t \in [t_0, t_0 + a)$, and $(t, v(t)) \in E$. If $v(t)$ satisfies the differential inequality

$$v'_+(t) < g(t, v(t)), \qquad t \in [t_0, t_0 + a),$$

it is said to be an *under-function* with respect to the initial value problem (1.2.1). On the other hand, if

$$v'_+(t) > g(t, v(t)), \qquad t \in [t_0, t_0 + a),$$

$v(t)$ is said to be an *over-function.*

A fundamental result on scalar differential inequalities is the following:

THEOREM 1.2.1. Let E be an open (t, u)-set in R^2 and $g \in C[E, R]$. Assume that $v, w \in C[[t_0, t_0 + a), R]$ and $(t, v(t))$, $(t, w(t)) \in E$, $t \in [t_0, t_0 + a)$. Suppose further that

$$v(t_0) < w(t_0), \tag{1.2.2}$$

and, for $t \in (t_0, t_0 + a)$, the inequalities

$$D_-v(t) \leqslant g(t, v(t)), \tag{1.2.3}$$

$$D_-w(t) > g(t, w(t)) \tag{1.2.4}$$

hold. Then,

$$v(t) < w(t), \qquad t \in [t_0, t_0 + a). \tag{1.2.5}$$

Proof. If assertion (1.2.5) is false, then the set

$$Z = [t \in [t_0, t_0 + a): w(t) \leqslant v(t)]$$

is nonempty. Defining $t_1 = \inf Z$, it is clear from (1.2.2) that $t_0 < t_1$. Furthermore,

$$v(t_1) = w(t_1) \tag{1.2.6}$$

and

$$v(t) < w(t), \qquad t \in [t_0, t_1). \tag{1.2.7}$$

Using (1.2.6) and (1.2.7), we obtain, for small $h < 0$,

$$\frac{v(t_1 + h) - v(t_1)}{h} > \frac{w(t_1 + h) - w(t_1)}{h},$$

which in its turn implies

$$D_-v(t_1) \geqslant D_-w(t_1). \tag{1.2.8}$$

The inequalities (1.2.3), (1.2.4), and (1.2.8) together with (1.2.6) lead us to the contradiction

$$g(t_1, v(t_1)) > g(t_1, w(t_1)).$$

Hence Z is empty, and the statement (1.2.5) follows.

REMARK 1.2.1. It is obvious from the proof that the inequalities (1.2.3) and (1.2.4) can also be replaced by

$$D_-v(t) < g(t, v(t)),$$

$$D_-w(t) \geqslant g(t, w(t)),$$

respectively.

Note that the proof does not demand the validity of the inequalities (1.2.3) and (1.2.4) for all $t \in (t_0, t_0 + a)$. The following refinement is a consequence of this observation.

THEOREM 1.2.2. Let the assumptions of Theorem 1.2.1 hold, except that the inequalities (1.2.3) and (1.2.4) are satisfied for $t \in Z_1 = [t \in (t_0, t_0 + a) : v(t) = w(t)]$. Then (1.2.5) remains valid.

In fact, Theorem 1.2.1 can be subjected to further refinements. To this end, we require the following simple lemmas. Although we state them for scalar functions, it is easy to see that they are true for vector functions as well. Unless otherwise specified, let S denote an at-most countable subset of $[t_0, t_0 + a)$.

LEMMA 1.2.1. (Zygmund). Suppose that $u \in C[[t_0, t_0 + a), R]$ and the inequality $Du(t) \leqslant 0$ for $t \in [t_0, t_0 + a) - S$, D being a fixed Dini derivative. Then, $u(t)$ is nonincreasing in t on $[t_0, t_0 + a)$.

LEMMA 1.2.2. Let $v, w \in C[[t_0, t_0 + a), R]$, and for some fixed Dini derivative $Dv(t) \leqslant w(t)$ for $t \in [t_0, t_0 + a) - S$. Then, $D_{-}v(t) \leqslant w(t)$ for $t \in [t_0, t_0 + a)$.

Proof. Define the function

$$m(t) = v(t) - \int_{t_0}^{t} w(s)\, ds.$$

It then follows, from the assumption, that

$$Dm(t) = Dv(t) - w(t) \leqslant 0, \qquad t \in [t_0, t_0 + a) - S.$$

Hence, by Lemma 1.2.1, $m(t)$ is nonincreasing in t on $[t_0, t_0 + a)$. Consequently,

$$D_{-}m(t) = D_{-}v(t) - w(t) \leqslant 0, \qquad t \in [t_0, t_0 + a),$$

and the lemma is proved.

REMARK 1.2.2. In the light of Lemma 1.2.2, it is clear that Theorem 1.2.1 remains true when the inequalities (1.2.3) and (1.2.4) hold for $t \in [t_0, t_0 + a) - S$, D being any fixed Dini derivative.

It will now be shown that any solution of the initial value problem (1.2.1) can be bracketed between its under- and over-functions.

THEOREM 1.2.3. Let $v(t)$, $w(t)$ be under- and over-functions with respect to the initial value problem (1.2.1), respectively, on $[t_0, t_0 + a)$. If $u(t)$ is any solution of (1.2.1) existing on $[t_0, t_0 + a)$ such that

$$v(t_0) = u_0 = w(t_0), \tag{1.2.9}$$

then

$$v(t) < u(t) < w(t), \qquad t \in [t_0, t_0 + a). \tag{1.2.10}$$

Proof. We shall prove the right half of the inequality (1.2.10). Similar reasoning can be used for the left half. Let $w(t)$ and $u(t)$ be an over-function and a solution of (1.2.1), respectively. Let $m(t) = w(t) - u(t)$. Then, $m'_+(t_0) > 0$ because of (1.2.9). It follows that $m(t)$ is increasing to the right of t_0 in a sufficiently small interval $t_0 \leqslant t \leqslant t_0 + \epsilon$, which implies that

$$u(t_0 + \epsilon) < w(t_0 + \epsilon).$$

Furthermore,

$$u'(t) \leqslant g(t, u(t))$$

and

$$w'_+(t) > g(t, w(t))$$

for $t \in [t_0 + \epsilon, t_0 + a)$. A direct application of Theorem 1.2.1 yields that

$$u(t) < w(t), \qquad t \in [t_0, t_0 + a).$$

This proves the theorem.

COROLLARY 1.2.1. Let E be an open (t, u)-set in R^2, $g_1, g_2 \in C[E, R]$, and

$$g_1(t, u) < g_2(t, u), \qquad (t, u) \in E.$$

Let $u_1(t)$, $u_2(t)$ be any two solutions of

$$u_1' = g_1(t, u), \qquad u_2' = g_2(t, u),$$

respectively, existing on $[t_0, t_0 + a)$ such that $u_1(t_0) < u_2(t_0)$. Then $u_1(t) < u_2(t)$, $t \in [t_0, t_0 + a)$.

COROLLARY 1.2.2. Let E be an open (t, u, v)-set in R^3, and $g \in C[E, R]$, and $g(t, u, v)$ is nondecreasing in v for each fixed t and u. Let $u, v \in C[[t_0, t_0 + a), R]$ such that $u'(t)$, $v'(t)$ exist, $(t, u(t), u'(t))$, $(t, v(t), v'(t)) \in E$ for $t \in [t_0, t_0 + a)$. Assume that the inequalities

$$g(t, u(t), u'(t)) \leqslant 0,$$

$$g(t, v(t), v'(t)) > 0$$

hold for $t \in [t_0, t_0 + a)$. Then, $u(t_0) < v(t_0)$ implies $u(t) < v(t)$ for $t \in [t_0, t_0 + a)$.

1.3. Maximal and minimal solutions

The notion of maximal and minimal solutions of (1.2.1) will now be introduced.

DEFINITION 1.3.1. Let $r(t)$ be a solution of the scalar differential equation (1.2.1) on $[t_0, t_0 + a)$. Then $r(t)$ is said to be a *maximal solution* of (1.2.1) if, for every solution $u(t)$ of (1.2.1) existing on $[t_0, t_0 + a)$, the inequality

$$u(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a) \tag{1.3.1}$$

holds. A *minimal solution* $\rho(t)$ may be defined similarly by reversing the inequality (1.3.1).

We shall now consider the existence of maximal and minimal solutions of (1.2.1) under the hypothesis of Peano's existence theorem.

THEOREM 1.3.1. Let $g \in C[R_0, R]$, where R_0 is the rectangle $t_0 \leqslant t \leqslant t_0 + a$, $|u - u_0| \leqslant b$, and $|g(t, u)| \leqslant M$ on R_0. Then there exist a maximal solution and a minimal solution of (1.2.1) on $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, b/(2M + b))$.

Proof. We shall prove the existence of the maximal solution only, since the case of the minimal solution is very similar. Let $0 < \epsilon \leqslant b/2$. Consider the differential equation with an initial condition

$$u' = g(t, u) + \epsilon, \qquad u(t_0) = u_0 + \epsilon. \tag{1.3.2}$$

Observing that

$$g_\epsilon(t, u) = g(t, u) + \epsilon$$

is defined and continuous on

$$R_\epsilon : t_0 \leqslant t \leqslant t_0 + a, \qquad |u - (u_0 + \epsilon)| \leqslant b/2,$$

$R_\epsilon \subset R_0$ and $|g_\epsilon(t, u)| \leqslant M + (b/2)$ on R_ϵ, we deduce from Theorem 1.1.2 that the initial value problem (1.3.2) has a solution $u(t, \epsilon)$ on the interval $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, b/(2M + b))$. For $0 < \epsilon_2 < \epsilon_1 \leqslant \epsilon$, we have

$$u(t_0, \epsilon_2) < u(t_0, \epsilon_1),$$

$$u'(t, \epsilon_2) \leqslant g(t, u(t, \epsilon_2)) + \epsilon_2,$$

$$u'(t, \epsilon_1) > g(t, u(t, \epsilon_1)) + \epsilon_2, \qquad t \in [t_0, t_0 + \alpha].$$

We can apply Theorem 1.2.1 to get

$$u(t, \epsilon_2) < u(t, \epsilon_1), \qquad t \in [t_0, t_0 + \alpha].$$

Since the family of functions $u(t, \epsilon)$ is equicontinuous and uniformly bounded on $[t_0, t_0 + \alpha]$, it follows by Theorem 1.1.1 that there exists a decreasing sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$, and the uniform limit

$$r(t) = \lim_{n\to\infty} u(t, \epsilon_n)$$

exists on $[t_0, t_0 + \alpha]$. Clearly, $r(t_0) = u_0$. The uniform continuity of g implies that $g(t, u(t, \epsilon_n))$ tends uniformly to $g(t, r(t))$ as $n \to \infty$, and thus term-by-term integration is applicable to

$$u(t, \epsilon_n) = u_0 + \epsilon_n + \int_{t_0}^{t} g(s, u(s, \epsilon_n))\, ds,$$

which in turn shows that the limit $r(t)$ is a solution of (1.2.1) on $[t_0, t_0 + \alpha]$.

We shall now show that $r(t)$ is the desired maximal solution of (1.2.1) on $[t_0, t_0 + \alpha]$ satisfying (1.3.1). Let $u(t)$ be any solution of (1.2.1) existing on $[t_0, t_0 + \alpha]$. Then,

$$u(t_0) = u_0 < u_0 + \epsilon = u(t_0, \epsilon),$$

$$u'(t) < g(t, u(t)) + \epsilon,$$

$$u'(t, \epsilon) \geqslant g(t, u(t, \epsilon)) + \epsilon,$$

for $t \in [t_0, t_0 + \alpha]$ and $\epsilon \leqslant b/2$. By Remark 1.2.1, we obtain that

$$u(t) < u(t, \epsilon), \qquad t \in [t_0, t_0 + \alpha].$$

The uniqueness of the maximal solution shows that $u(t, \epsilon)$ tends uniformly to $r(t)$ on $[t_0, t_0 + \alpha]$ as $\epsilon \to 0$. This proves the theorem.

This existence theorem, together with the extension Theorem 1.1.3, implies the following:

THEOREM 1.3.2. Let $g \in C[E, R]$, where E is an open (t, u)-set in R^2 and $(t_0, u_0) \in E$. Then (1.2.1) has maximal and minimal solutions that can be extended to the boundary of E.

The lemmas given below are useful in certain later applications.

LEMMA 1.3.1. Let the hypothesis of Theorem 1.3.2 hold, and let $[t_0, t_0 + a)$ be the largest interval of existence of the maximal solution

$r(t)$ of (1.2.1). Suppose $[t_0, t_1]$ is a compact subinterval of $[t_0, t_0 + a)$. Then there is an $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, the maximal solution $r(t, \epsilon)$ of Eq. (1.3.2) exists over $[t_0, t_1]$, and

$$\lim_{\epsilon \to 0} r(t, \epsilon) = r(t)$$

uniformly on $[t_0, t_1]$.

Proof. Let E_0 be an open bounded set, $\bar{E}_0 \subset E$, and $(t, r(t)) \in E$ for $t \in [t_0, t_1]$. We can choose a $b > 0$ such that, for $t \in [t_0, t_1]$, the rectangle

$$R_t^\epsilon\colon [t, t + b], \qquad |u - (r(t) + \epsilon)| \leqslant b,$$

is included in E_0 for $\epsilon \leqslant b/2$. Let $|g(t, u)| \leqslant M$ on E_0. Then it is evident that

$$|g(t, u) + \epsilon| \leqslant M + b/2$$

on R_t^ϵ, for $t \in [t_0, t_1]$ and $0 < \epsilon \leqslant b/2$. Consider the rectangle $R_{t_0}^\epsilon$. It follows from Theorem 1.3.1 that the maximal solution $r(t, \epsilon)$ of (1.3.2) exists on $[t_0, t_0 + \eta]$, $\eta = \min(b, 2b/(2M + b))$. Note that η does not depend upon ϵ. Furthermore, proceeding as in Theorem 1.3.1, we can conclude, in view of the uniqueness of the maximal solution $r(t)$ of (1.2.1), that

$$\lim_{\epsilon \to 0} r(t, \epsilon) = r(t)$$

uniformly on $[t_0, t_0 + \eta]$. This implies that

$$\lim_{\epsilon \to 0} r(t_0 + \eta, \epsilon) = r(t_0 + \eta).$$

Consequently, there is an $\epsilon_1 \leqslant b/2$ such that, for $0 < \epsilon \leqslant \epsilon_1$, we have

$$r(t_0 + \eta, \epsilon) \leqslant r(t_0 + \eta) + \epsilon.$$

We can now repeat the foregoing argument with respect to the rectangle $R_{t_0+\eta}^\epsilon$, $\epsilon < \epsilon_1$, to show that there exists an $\epsilon_2 < \epsilon_1$ such that, for $\epsilon < \epsilon_2$, the maximal solution $\hat{r}(t, \epsilon)$ of

$$u' = g(t, u) + \epsilon, \qquad u(t_0 + \eta) = r(t_0 + \eta) + \epsilon$$

exists on $[t_0 + \eta, t_0 + 2\eta]$, and

$$\lim_{\epsilon \to 0} \hat{r}(t, \epsilon) = r(t)$$

uniformly on $[t_0 + \eta, t_0 + 2\eta]$. For $\epsilon < \epsilon_2$, we can extend the function $r(t, \epsilon)$ by defining

$$r(t, \epsilon) = \hat{r}(t, \epsilon), \qquad t \in [t_0 + \eta, t_0 + 2\eta].$$

It is clear that $r(t, \epsilon)$ is the maximal solution of (1.3.2) on $[t_0, t_0 + 2\eta]$, and

$$\lim_{\epsilon\to 0} r(t, \epsilon) = r(t)$$

uniformly on $[t_0, t_0 + 2\eta]$.

By induction, it can be shown that there is an $\epsilon_0 = \epsilon_n$ such that $[t_0, t_1] \subset [t_0, t_1 + n\eta]$, that the maximal solution $r(t, \epsilon)$ of (1.3.2) exists on $[t_0, t_0 + n\eta]$ for $0 < \epsilon < \epsilon_0$, and that

$$\lim_{\epsilon\to 0} r(t, \epsilon) = r(t)$$

uniformly on $[t_0, t_0 + n\eta]$. The lemma is thus proved.

LEMMA 1.3.2. Let $g \in C[[t_0, t_0 + a] \times R, R]$ and nondecreasing in u for each $t \in [t_0, t_0 + a]$. Assume that

$$g(t, 0) \equiv 0, \tag{1.3.3}$$

$$|g(t, u)| \leqslant M \qquad \text{on} \quad [t_0, t_0 + a] \times R, \tag{1.3.4}$$

and $u(t) \equiv 0$ is the unique solution of

$$u' = g(t, u), \qquad u(t_0) = 0 \tag{1.3.5}$$

on $[t_0, t_0 + a]$. Then, the successive approximations

$$u_0(t) = M(t - t_0),$$

$$u_{n+1}(t) = \int_{t_0}^{t} g(s, u_n(s))\, ds$$

are well defined;

$$0 \leqslant u_{n+1}(t) \leqslant u_n(t) \qquad \text{on} \quad [t_0, t_0 + a], \tag{1.3.6}$$

and

$$\lim_{n\to\infty} u_n(t) \equiv 0 \qquad \text{uniformly on} \quad [t_0, t_0 + a]. \tag{1.3.7}$$

Moreover, for every $n \geqslant 1$, the maximal solution $r_n(t)$ of

$$u' = g(t, u) + kg(t, u_{n-1}(t)), \qquad u_n(t_0) = 0, \qquad k > 0,$$

exists on $[t_0, t_0 + a]$, and

$$\lim_{n\to\infty} r_n(t) \equiv 0 \qquad \text{uniformly on} \quad [t_0, t_0 + a]. \tag{1.3.8}$$

Proof. An easy induction proves (1.3.6). Since, by (1.3.4), $|u_n'(t)| \leqslant M$, using Theorem 1.1.1, we can conclude that $\lim_{n\to\infty} u_n(t) = u(t)$ uniformly

on $[t_0, t_0 + a]$. It is clear that $u(t)$ satisfies $u'(t) = g(t, u(t))$ and $u(t_0) = 0$. By (1.3.5), it follows that $u(t) \equiv 0$, and (1.3.7) is proved.

Given $\epsilon > 0$, there is an $n \geqslant n(\epsilon)$ such that

$$| kg(t, u_{n-1}(t)| < \epsilon, \qquad t \in [t_0, t_0 + a],$$

because of (1.3.3) and (1.3.7). Now an argument similar to that of Lemma 1.3.1 proves (1.3.8).

1.4. Comparison theorems

An important technique in the theory of differential equations is concerned with estimating a function satisfying a differential inequality by the extremal solutions, of the corresponding differential equation. One of the results that is widely used is the following comparison theorem:

THEOREM 1.4.1. Let E be an open (t, u)-set in R^2 and $g \in C[E, R]$. Suppose that $[t_0, t_0 + a)$ is the largest interval in which the maximal solution $r(t)$ of (1.2.1) exists. Let $m \in C[(t_0, t_0 + a), R]$, $(t, m(t)) \in E$ for $t \in [t_0, t_0 + a)$, $m(t_0) \leqslant u_0$, and for a fixed Dini derivative,

$$Dm(t) \leqslant g(t, m(t)), \tag{1.4.1}$$

$t \in [t_0, t_0 + a) - S$. Then,

$$m(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a). \tag{1.4.2}$$

Proof. From Lemma 1.2.2, it follows that (1.4.1) can be replaced by

$$D_{-}m(t) \leqslant g(t, m(t)), \qquad t \in (t_0, t_0 + a). \tag{1.4.3}$$

Let $t_0 < \tau < t_0 + a$. By Lemma 1.3.1, the maximal solutions $r(t, \epsilon)$ of (1.3.2) exist on $[t_0, \tau]$ for all $\epsilon > 0$ sufficiently small, and

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon) \tag{1.4.4}$$

uniformly on $[t_0, \tau]$. Using (1.3.2) and (1.4.3) and applying Theorem 1.2.1, we derive that

$$m(t) < r(t, \epsilon), \qquad t \in [t_0, \tau]. \tag{1.4.5}$$

The last inequality, together with (1.4.4), proves the assertion of the theorem.

REMARK 1.4.1. If the inequality (1.4.1) is reversed and $m(t_0) \geqslant u_0$, then we have to replace the conclusion (1.4.2) by $m(t) \geqslant \rho(t)$, where $\rho(t)$ is the minimal solution of (1.2.1).

Theorem 1.4.1 can also be proved under a weaker hypothesis.

THEOREM 1.4.2. Let $m(t)$, $r(t)$ be as in Theorem 1.4.1, and

$$\hat{Z} = [t \in [t_0, t_0 + a) : r(t) < m(t) < r(t) + \epsilon_0], \tag{1.4.6}$$

for some $\epsilon_0 > 0$. If (1.4.1) is satisfied for $t \in \hat{Z} - \hat{S}$, where $\hat{S}$ is an at-most countable subset of $\hat{Z}$, then (1.4.2) holds.

Proof. It is enough to prove (1.4.5). As before, Lemma 1.2.2 implies that (1.4.3) is satisfied for $t \in \hat{Z}$. Proceeding as in the proof of Theorem 1.2.1, we arrive at a t_1 such that

$$m(t_1) = r(t_1, \epsilon).$$

In view of (1.4.4), there exists an $\epsilon_0 > 0$ such that

$$r(t, \epsilon) < r(t) + \epsilon_0, \qquad t \in [t_0, \tau].$$

Moreover, we have $r(t) < r(t, \epsilon)$, and hence there results the inequality

$$r(t) < r(t, \epsilon) < r(t) + \epsilon_0, \qquad t \in [t_0, \tau].$$

It therefore follows from (1.4.6) that

$$r(t_1) < m(t_1) < r(t_1) + \epsilon_0,$$

which implies that $t_1 \in \hat{Z}$. Hence, (1.4.3) is satisfied for such a t_1, and this is sufficient to establish the desired result.

We now give a modification of Theorem 1.2.1. It is evident that the proof of Theorem 1.2.1 breaks down if we do not assume one of the relations (1.2.3) and (1.2.4) to be a strict inequality. This, however, can be relaxed provided g satisfies a further assumption.

THEOREM 1.4.3. Let the hypothesis of Theorem 1.2.1 hold except that the inequalities (1.2.3) and (1.2.4) are replaced by

$$D_{-}v(t) \leqslant g(t, v(t)), \tag{1.4.7}$$

$$D_{-}w(t) \geqslant g(t, w(t)) \tag{1.4.8}$$

for $t \in (t_0, t_0 + a)$. Assume further that, for each $\tau \in (t_0, t_0 + a)$ and $t \in [t_0, \tau]$, g satisfies the condition

$$g(t, u_1) - g(t, u_2) \geqslant -G(\tau + t_0 - t, u_1 - u_2), \qquad u_1 \geqslant u_2, \tag{1.4.9}$$

where $G \in C[[t_0, t_0 + a) \times R, R]$, and $r(t) \equiv 0$ is the maximal solution of

$$u' = G(t, u), \qquad u(t_0) = 0.$$

Then (1.2.5) holds.

Proof. Proceeding as in the proof of Theorem 1.2.1, there exists a $t_1 \in (t_0, t_0 + a)$ such that

$$v(t_1) = w(t_1), \tag{1.4.10}$$

and

$$v(t) < w(t), \qquad t_0 \leqslant t < t_1. \tag{1.4.11}$$

Define $v_1(t) = v(t_1 + t_0 - t)$ and $w_1(t) = w(t_1 + t_0 - t)$. This, in view of (1.4.10) and (1.4.11), yields that

$$v_1(t_0) = w_1(t_0): \tag{1.4.12}$$

$$v_1(t) \leqslant w_1(t), \qquad t \in [t_0, t_1]. \tag{1.4.13}$$

Setting $m(t) = w_1(t) - v_1(t)$, the definitions of v_1, w_1 and the assumptions (1.4.7) and (1.4.8) imply the inequality

$$D_-m(t) = D_-w_1(t) - D_-v_1(t) \leqslant g_1(t, w_1(t)) - g_1(t, v_1(t)),$$

where $g_1(t, u) = -g(t_1 + t_0 - t, u)$. Since (1.4.13) holds, we can use (1.4.9) to arrive at

$$D_-m(t) \leqslant G(t, m(t)), \qquad t \in [t_0, t_1].$$

By Theorem 1.4.1, we have

$$m(t) \leqslant r(t), \qquad t \in [t_0, t_1], \tag{1.4.14}$$

where $r(t)$ is the maximal solution of $u' = G(t, u)$, such that $r(t_0) = m(t_0)$. From the definition of $m(t)$ and (1.4.12) and (1.4.13), we deduce that $m(t) \geqslant 0$, $t \in [t_0, t_1]$, and $m(t_0) = 0$. Then, the inequality (1.4.14) and the assumption $r(t) \equiv 0$ show that

$$v(t) = w(t), \qquad t \in [t_0, t_1],$$

which, however, is contrary to the assumption (1.4.11) and the definition of t_1. Hence, the set Z is empty, and the theorem is proved.

To give another comparison theorem that, in certain situations, is more useful than Theorem 1.4.1, we require the following result:

THEOREM 1.4.4. Let E be the product space $[t_0, t_0 + a) \times R^2$ and $g \in C[E, R]$. Assume that g is nondecreasing in v for each t and u. Suppose that $r(t)$ is the maximal solution of the differential equation

$$u' = g(t, u, u), \qquad u(t_0) = u_0 \geqslant 0 \tag{1.4.15}$$

existing on $[t_0, t_0 + a)$, and

$$r(t) \geqslant 0, \qquad t \in [t_0, t_0 + a). \tag{1.4.16}$$

Then, the maximal solution $r_1(t)$ of

$$u' = g_1(t, u), \qquad u(t_0) = u_0 \geqslant 0, \tag{1.4.17}$$

where $g_1(t, u) = g(t, u, r(t))$, exists on $[t_0, t_0 + a)$ and

$$r(t) = r_1(t), \qquad t \in [t_0, t_0 + a). \tag{1.4.18}$$

Proof. By Theorems 1.3.1 and 1.3.2, the maximal solution $r_1(t)$ of (1.4.17) exists on an interval $[t_0, t_0 + \alpha]$, $\alpha < a$, which can be extended to the boundary of E. This implies that either $r_1(t)$ is defined over $[t_0, t_0 + a)$ or there exists a $t_1 < t_0 + a$ such that

$$| r_1(t_k)| \to \infty \tag{1.4.19}$$

for a certain sequence $\{t_k\}$, $t_k \to t_1^-$ as $k \to \infty$. Observe that

$$r'(t) = g(t, r(t), r(t)) = g_1(t, r(t)),$$

and this yields, from Theorem 1.4.1, that

$$r(t) \leqslant r_1(t), \tag{1.4.20}$$

as far as $r_1(t)$ exists. It follows from (1.4.16), (1.4.19), and (1.4.20) that

$$r_1(t_k) \to +\infty \tag{1.4.21}$$

as $t_k \to t_1^-$.

We shall show that (1.4.21) cannot be true. For this purpose, consider the maximal solution $r(t, \epsilon)$ of

$$u' = g(t, u, u) + \epsilon, \qquad u(t_0) = u_0 + \epsilon, \qquad u_0 \geqslant 0, \tag{1.4.22}$$

which, by Lemma 1.3.1, exists on $[t_0, t_1 + \nu]$, $\nu > 0$, and $t_1 + \nu < t_0 + a$, for sufficiently small $\epsilon > 0$. Moreover, we have from (1.4.22) that

$$r'(t, \epsilon) > g(t, r(t, \epsilon), r(t, \epsilon)), \tag{1.4.23}$$

and

$$r(t_0) < r(t_0, \epsilon).$$

Hence, one gets, from Theorem 1.2.1, the inequality

$$r(t) < r(t, \epsilon), \qquad t \in [t_0, t_1 + \nu]. \tag{1.4.24}$$

Since g is nondecreasing in v, (1.4.23) and (1.4.24) lead to

$$r'(t, \epsilon) > g_1(t, r(t, \epsilon)), \qquad t \in [t_0, t_1 + \nu].$$

But $r_1'(t) = g(t, r_1(t), r(t)) = g_1(t, r_1(t))$, $t \in [t_0, t_1)$, and $r_1(t_0) < r_1(t_0, \epsilon)$. An application of Theorem 1.2.1 again shows that

$$r_1(t) < r(t, \epsilon), \qquad t \in [t_0, t_1). \tag{1.4.25}$$

Since $r(t, \epsilon)$ exists on $[t_0, t_1 + \nu]$, $\nu > 0$, (1.4.21) leads us to a contradiction because of (1.4.25), and this proves the existence of $r_1(t)$ on $[t_0, t_0 + a)$.

To prove (1.4.18), we now see that (1.4.20) is true for $t \in [t_0, t_0 + a)$. Furthermore,

$$r_1'(t) = g_1(t, r_1(t)) = g(t, r_1(t), r(t)).$$

From the monotonic character of g in v and (1.4.20), one gets

$$r_1'(t) \leqslant g(t, r_1(t), r_1(t)).$$

Theorem 1.4.1 now gives that

$$r_1(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a).$$

This inequality, together with (1.4.20), proves (1.4.18), as is desired.

THEOREM 1.4.5. *Let the hypothesis of Theorem 1.4.4 hold; $m \in C[[t_0, t_0 + a), R]$ such that $(t, m(t), v) \in E$, $t \in [t_0, t_0 + a)$, and $m(t_0) \leqslant u_0$. Assume that for a fixed Dini derivative the inequality*

$$Dm(t) \leqslant g(t, m(t), v) \tag{1.4.26}$$

is satisfied for $t \in [t_0, t_0 + a) - S$. Then, for all $v \leqslant r(t)$, $t \in [t_0, t_0 + a)$, we have

$$m(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a). \tag{1.4.27}$$

Proof. Let $v \leqslant r(t)$, $t \in [t_0, t_0 + a)$. Then, using the monotonicity of g in v, the inequality (1.4.26) reduces to

$$Dm(t) \leqslant g_1(t, m(t)), \qquad t \in [t_0, t_0 + a) - S,$$

where $g_1(t, m(t)) = g(t, m(t), r(t))$. If $r_1(t)$ is the maximal solution of (1.4.17), Theorem 1.4.4 shows that $r_1(t)$ exists on $[t_0, t_0 + a)$ and (1.4.18) is true. Now a straightforward application of Theorem 1.4.1 assures the inequality (1.4.27).

COROLLARY 1.4.1. Let E be an open (t, u)-set and $g \in C[E, R]$. Let $m \in C[[t_0, t_0 + a), R]$, $(t, m(t)) \in E$, and

$$D\_m(t) \leqslant g(t, m(t)), \qquad t \in (t_0, t_0 + a).$$

Assume that, for each $\tau \in (t_0, t_0 + a)$ and $t \in [t_0, \tau]$, g satisfies the condition

$$g(t, u_1) - g(t, u_2) \geqslant -G(\tau + t_0 - t, u_1 - u_2), \qquad u_1 \geqslant u_2,$$

where $G \in C[[t_0, t_0 + a) \times R, R]$, and $r(t) \equiv 0$ is the maximal solution of

$$u' = G(t, u), \qquad u(t_0) = 0.$$

Then $m(t_0) < u_0$ implies $m(t) < u(t)$, $t \in [t_0, t_0 + a)$, where $u(t)$ is any solution of $u' = g(t, u)$, $u(t_0) = u_0$, existing on $[t_0, t_0 + a)$.

The maximal and minimal solutions may be defined to the left of t_0, and their existence may be proved using the previous arguments with necessary modifications. A result parallel to Theorem 1.4.1, concerning the minimal solution to the left, is useful in later applications. We shall state this as a theorem, omitting its proof.

THEOREM 1.4.6. Let E be an open (t, u)-set in R^2 and $g \in C[E, R]$. Suppose that $m \in C[(t_0 - a, t_0], R]$, $(t, m(t)) \in E$ for $t \in (t_0 - a, t_0]$, $m(t_0) \geqslant u_0$, and for any fixed Dini derivative

$$Dm(t) \leqslant g(t, m(t)), \qquad t \in (t_0 - a, t_0).$$

Then

$$m(t) \geqslant \rho(t),$$

as far as $\rho(t)$ exists to the left of t_0, $\rho(t)$ being the left minimal solution of (1.2.1).

1.5. Finite systems of differential inequalities

Many of the results considered so far for scalar differential inequalities will now be extended, in the sections that follow, to finite systems of differential inequalities. To avoid repetition, let us agree on the following: the subscript i ranges over the integers $1, 2,..., n$; let $0 \leqslant k \leqslant n$; the subscripts p and q range over the integers $1, 2,..., k$ and $k + 1, k + 2,..., n$, respectively. We shall be using vectorial inequalities freely, with the understanding that the same inequalities hold between their corresponding components.

We shall consider the differential system with an initial condition, written in the vectorial form

$$u' = g(t, u), \qquad u(t_0) = u_0\,, \tag{1.5.1}$$

where $g \in C[E, R^n]$ and E is an open (t, u)-set in R^{n+1}.

DEFINITION 1.5.1. Let $v \in C[[t_0\,, t_0 + a), R^n]$; $(t, v(t)) \in E$, and $v'_+(t)$ exists for $t \in [t_0\,, t_0 + a)$. The function $v(t)$ is said to be a *k under* $(n - k)$ *over-function* with respect to the initial value problem (1.5.1) if

$$v'_{p,+}(t) < g_p(t, v(t)),$$

$$v'_{q,+}(t) > g_q(t, v(t))$$

hold for $t \in [t_0\,, t_0 + a)$. If $v(t)$ satisfies the reversed inequalities, it is said to be a *k over* $(n - k)$ *under-function*.

These definitions clearly include the definitions of under- and over-functions as special cases, viz., $k = 0$ or $k = n$.

We require that the function $g(t, u)$ should satisfy certain monotonic properties, which are listed below.

DEFINITION 1.5.2. The function $g(t, u)$ is said to possess a *mixed quasimonotone property* if the following conditions hold:

(*i*) $g_p(t, u)$ is nondecreasing in u_j, $j = 1, 2,..., k$, $j \neq p$, and nonincreasing in u_q.

(ii) $g_q(t, u)$ is nonincreasing in u_p and nondecreasing in u_j, $j = k + 1, k + 2,..., n$, $j \neq q$.

Evidently, the particular cases $k = n$ and $k = 0$ in the mixed quasimonotone property correspond to quasi-monotone nondecreasing and quasi-monotone nonincreasing properties of the function $g(t, u)$, respectively. Furthermore, $g(t, u)$ is said to possess mixed monotone property if, in conditions (i) and (ii), $j \neq p$, $j \neq q$ are not demanded.

An extension of Theorem 1.2.1 which plays an equally important role is the following:

THEOREM 1.5.1. Let (i) $g \in C[E, R^n]$, where E is an open (t, u)-set in R^{n+1}; (ii) $v, w \in C[[t_0, t_0 + a), R^n]$, $(t, v(t))$, $(t, w(t)) \in E$ for $t \in [t_0, t_0 + a)$; and (iii) $g(t, u)$ possess a mixed quasi-monotone property. Assume further that

$$v_p(t_0) < w_p(t_0), \qquad v_q(t_0) > w_q(t_0), \tag{1.5.2}$$

and, for $t \in (t_0, t_0 + a)$, the inequalities

$$D_-v_p(t) \leqslant g_p(t, v(t)), \tag{1.5.3}$$

$$D_-v_q(t) > g_q(t, v(t)), \tag{1.5.4}$$

$$D_-w_p(t) > g_p(t, w(t)), \tag{1.5.5}$$

$$D_-w_q(t) \leqslant g_q(t, w(t)) \tag{1.5.6}$$

are satisfied. Then,

$$v_p(t) < w_p(t), \qquad v_q(t) > w_q(t), \qquad t \in [t_0, t_0 + a). \tag{1.5.7}$$

Proof. Define $m_p(t) = w_p(t) - v_p(t)$ and $m_q(t) = v_q(t) - w_q(t)$. Then, because of (1.5.2),

$$m_i(t_0) > 0, \qquad i = 1, 2, \dots, n. \tag{1.5.8}$$

Suppose that the assertion (1.5.7) is not true. Then, the set

$$Z = \bigcup_{i=1}^{n} [t \in [t_0, t_0 + a): m_i(t) \leqslant 0]$$

is nonempty. Let $t_1 = \inf Z$. By (1.5.8), it is obvious that $t_1 > t_0$. Since the set Z is closed, $t_1 \in Z$, and consequently there exists a j such that

$$m_j(t_1) = 0. \tag{1.5.9}$$

If (1.5.9) is not true, one would have $m_j(t_1) < 0$, which implies $m_j(t) < 0$ in a sufficiently small neighborhood to the left of t_1. This contradicts the definition of t_1, and therefore (1.5.9) is valid. Moreover,

$$m_i(t_1) \geqslant 0, \qquad i \neq j, \tag{1.5.10}$$

and

$$D_-m_j(t_1) \leqslant 0. \tag{1.5.11}$$

Suppose that $1 \leqslant j \leqslant k$. Then (1.5.11), along with (1.5.3) and (1.5.5), gives

$$g_j(t_1, w(t_1)) < g_j(t_1, v(t_1)). \tag{1.5.12}$$

The mixed quasi-monotone property of $g(t, u)$ in u, in view of (1.5.9) and (1.5.10), yields

$$g_j(t_1, v(t_1)) \leqslant g_j(t_1, w(t_1)). \tag{1.5.13}$$

The inequalities (1.5.12) and (1.5.13) lead us to a contradiction. If, on the other hand, $k + 1 \leqslant j \leqslant n$, arguing as before, we arrive at the contradiction

$$g_j(t_1, w(t_1)) > g_j(t_1, w(t_1)),$$

using the relations (1.5.4), (1.5.6), (1.5.9), (1.5.10), (1.5.11), and the mixed quasi-monotone property of $g(t, u)$ in u. Hence the set Z is empty, and (1.5.7) is proved.

COROLLARY 1.5.1. Let conditions (i), (ii), and (iii) of Theorem 1.5.1 be satisfied. Assume that, for $t \in (t_0, t_0 + a)$, the inequalities

$$D_-v(t) \leqslant g(t, v(t)),$$
$$D_-w(t) > g(t, w(t))$$

hold. Then, $v(t_0) < w(t_0)$ implies

$$v(t) < w(t), \qquad t \in [t_0, t_0 + a).$$

REMARK 1.5.1. Notice that the proof of Theorem 1.5.1 remains unchanged even when the inequalities (1.5.3)–(1.5.6) are replaced by

$$D_-v_p(t) < g_p(t, v(t)),$$
$$D_-v_q(t) \geqslant g_q(t, v(t)),$$
$$D_-w_p(t) \geqslant g_p(t, w(t)),$$
$$D_-w_q(t) < g_q(t, w(t)).$$

REMARK 1.5.2. One can, in Theorem 1.5.1 and the following corollary, use any fixed Dini derivative D in place of D_-, the corresponding inequalities being satisfied only for $t \in [t_0, t_0 + a) - S$. This follows from Lemmas 1.2.1 and 1.2.2.

The next theorem is an analog of Theorem 1.2.3.

THEOREM 1.5.2. Let $v(t)$, $w(t)$ be k under $(n - k)$ over-, k over $(n - k)$ under-functions, respectively, for $t \in [t_0, t_0 + a)$, with respect to the initial value problem (1.5.1). Assume that $g(t, u)$ has mixed quasi-monotone property. Let $u(t)$ be any solution of (1.5.1) existing on $[t_0, t_0 + a)$ such that

$$v(t_0) = u_0 = w(t_0). \tag{1.5.14}$$

Then

$$v_p(t) < u_p(t) < w_p(t), \tag{1.5.15}$$

$$v_q(t) > u_q(t) > w_q(t) \tag{1.5.16}$$

for $t \in [t_0, t_0 + a)$.

Proof. If (1.5.15) and (1.5.16) hold for $t_0 < t < \hat{t}_0$, $\hat{t}_0$ sufficiently close to t_0, then one can deduce the assertion of the theorem by the application of Theorem 1.5.1 and the subsequent Remark 1.5.1. Indeed, such a $\hat{t}_0$ exists. For, defining

$$m_p(t) = u_p(t) - v_p(t), \qquad m_q(t) = v_q(t) - u_q(t)$$

and noting that $m_i(t_0) = 0$ because of (1.5.14), it is easy to deduce that $m'_{i,+}(t_0) > 0$, which implies $m_i(t)$ is increasing in a sufficiently small neighborhood of t_0, say $t_0 \leqslant t \leqslant t_1$. Similar argument with

$$m_p^*(t) = w_p(t) - u_p(t), \qquad m_q^*(t) = u_q(t) - w_q(t)$$

shows that $m_i^*(t)$ increases in $t_0 \leqslant t \leqslant t_2$, t_2 being sufficiently close to t_0. Now, the desired $\hat{t}_0 = \min(t_1, t_2)$. The proof is therefore complete.

COROLLARY 1.5.2. Let $v(t)$, $w(t)$ be under- and over-functions, respectively, with respect to the initial value problem (1.5.1) for $t \in [t_0, t_0 + a)$. Assume that $g(t, u)$ is quasi-monotone nondecreasing in u. Let $u(t)$ be any solution of (1.5.1) existing on $[t_0, t_0 + a)$ such that

$$v(t_0) = u_0 = w(t_0).$$

Then,

$$v(t) < u(t) < w(t), \qquad t \in [t_0, t_0 + a).$$

COROLLARY 1.5.3. Let (i) $f, g \in C[E, R^n]$, where E is an open (t, u)-set in R^{n+1}; (ii) either f or g possess a mixed quasi-monotone property; (iii) $f_p(t, u) < g_p(t, u)$, $f_q(t, u) > g_q(t, u)$, $(t, u) \in E$; (iv) $u(t)$, $v(t)$ be any two solutions of $u' = f(t, u)$, $v' = g(t, v)$, existing on $[t_0, t_0 + a)$, respectively, such that $u_{0,p} < v_{0,p}$, $u_{0,q} > v_{0,q}$. Then

$$u_p(t) < v_p(t), \qquad u_q(t) > v_q(t), \qquad t \in [t_0, t_0 + a).$$

1.6. Minimax solutions

DEFINITION 1.6.1. Let $r(t)$ be a solution of the differential system (1.5.1) existing on $[t_0, t_0 + a)$ such that, for every solution $u(t)$ of (1.5.1) on $[t_0, t_0 + a)$, the inequalities

$$u_p(t) \leqslant r_p(t), \qquad u_q(t) \geqslant r_q(t), \qquad t \in [t_0, t_0 + a) \tag{1.6.1}$$

or

$$u_p(t) \geqslant r_p(t), \qquad u_q(t) \leqslant r_q(t), \qquad t \in [t_0, t_0 + a) \tag{1.6.2}$$

are satisfied. In case of (1.6.1), $r(t)$ is called a *k max* $(n - k)$ *mini-solution* of (1.5.1), whereas, in case of (1.6.2), it is said to be a *k mini* $(n - k)$ *max-solution*. In either case, $r(t)$ is said to be a *minimax solution*.

A k max $(n - k)$ mini-solution reduces to a maximal solution when $k = n$ and to a minimal solution when $k = 0$. Similarly, a k mini $(n - k)$ max-solution coincides with a minimal solution and a maximal solution when $k = n$ and $k = 0$, respectively.

As minimax solutions include both maximal and minimal solutions as special cases, we consider below the existence problem for minimax solutions only.

THEOREM 1.6.1. Let $g \in C[R_0, R^n]$, where

$$R_0 : t_0 \leqslant t \leqslant t_0 + a, \qquad \| u - u_0 \| \leqslant b,$$

and $\| g(t, u)\| \leqslant M$ on R_0. Assume further that $g(t, u)$ possesses a mixed quasi-monotone property. Then, there exists a k max $(n - k)$ mini- and a k mini $(n - k)$ max-solution of (1.5.1) on $[t_0, t_0 + \eta]$, where $\eta = \min(a, b/(2M + b))$.

Proof. Let $0 < \epsilon < b/2$. Consider the initial value problem

$$\left.\begin{aligned} u_p' &= g_p(t, u) + \epsilon, \qquad & u_p(t_0) &= u_{0,p} + \epsilon; \\ u_q' &= g_q(t, u) - \epsilon, \qquad & u_q(t_0) &= u_{0,q} - \epsilon. \end{aligned}\right\} \tag{1.6.3}$$

Observe that $g_\epsilon \in C[R_\epsilon, R^n]$, where

$$g_\epsilon(t, u) = g(t, u) \pm \epsilon,$$

$$R_\epsilon : [(t, u) \in R^{n+1} : t_0 \leqslant t \leqslant t_0 + a, \qquad \| u - (u_0 \pm \epsilon)\| \leqslant b/2],$$

and $R_\epsilon \subset R_0$. Also,

$$\| g_\epsilon(t, u)\| \leqslant M + (b/2) \qquad \text{on} \quad R_\epsilon .$$

It therefore follows from Peano's Theorem 1.1.2 that the initial value problem (1.6.3) has a solution $u(t, \epsilon)$ on the interval $[t_0, t_0 + \eta]$, $\eta = \min(a, b/(2M + b))$. Let $0 < \epsilon_2 < \epsilon_1 \leqslant \epsilon$. Then, we have

$$u_p(t_0, \epsilon_2) < u_p(t_0, \epsilon_1), \qquad u_q(t_0, \epsilon_2) > u_q(t_0, \epsilon_1);$$

$$u_p'(t, \epsilon_2) \leqslant g_p(t, u(t, \epsilon_2)) + \epsilon_2 ;$$

$$u_q'(t, \epsilon_2) \geqslant g_q(t, u(t, \epsilon_2)) - \epsilon_2 ;$$

$$u_p'(t, \epsilon_1) > g_p(t, u(t, \epsilon_1)) + \epsilon_2 ;$$

$$u_q'(t, \epsilon_1) < g_q(t, u(t, \epsilon_1)) - \epsilon_2 .$$

An application of Theorem 1.5.1 yields

$$u_p(t, \epsilon_2) < u_p(t, \epsilon_1), \qquad u_q(t, \epsilon_2) > u_q(t, \epsilon_1),$$

for $t \in [t_0, t_0 + \eta]$. Since the family of functions $u(t, \epsilon)$ is equicontinuous and uniformly bounded, one can establish that

$$\lim_{\epsilon_n \to 0} u(t, \epsilon_n) = r(t)$$

uniformly on $[t_0, t_0 + \eta]$ and that $r(t)$ is a solution of (1.5.1) on $[t_0, t_0 + \eta]$.

To show that $r(t)$ is a k max $(n - k)$ mini-solution of (1.5.1) on $[t_0, t_0 + \eta]$, we have to prove that (1.6.1) is satisfied. Let $u(t)$ be any solution of (1.5.1) existing on $[t_0, t_0 + \eta]$. Then,

$$u_p(t_0) < u_p(t_0, \epsilon), \qquad u_q(t_0) > u_q(t_0, \epsilon);$$

$$u_p'(t) < g_p(t, u(t)) + \epsilon;$$

$$u_q'(t) > g_q(t, u(t)) - \epsilon;$$

$$u_p'(t, \epsilon) \geqslant g_p(t, u(t, \epsilon)) + \epsilon;$$

$$u_q'(t, \epsilon) \leqslant g_q(t, u(t, \epsilon)) - \epsilon$$

for $\epsilon \leqslant b/2$. By Theorem 1.5.1, it follows that

$$u_p(t) < u_p(t, \epsilon),$$

$$u_q(t) > u_q(t, \epsilon)$$

for $t \in [t_0, t_0 + \eta]$. Consequently,

$$u_p(t) \leqslant \lim_{\epsilon \to 0} u_p(t, \epsilon) = r_p(t),$$

$$u_q(t) \geqslant \lim_{\epsilon \to 0} u_q(t, \epsilon) = r_q(t)$$

for $t \in [t_0, t_0 + \eta]$.

The existence of k mini $(n - k)$ max-solution can be established by changing the signs of ϵ in (1.6.3) and proceeding on similar lines. This proves the theorem.

This existence theorem for minimax solutions, together with the extension Theorem 1.1.3, implies the following:

THEOREM 1.6.2. Let hypotheses (i) and (iii) of Theorem 1.5.1 hold. Then, if $(t_0, u_0) \in E$, (1.5.1) has minimax solutions that can be extended to the boundary of E.

COROLLARY 1.6.1. Let the hypothesis of Theorem 1.6.2 hold. Let $[t_0, t_0 + a)$ be the largest interval of existence of k max $(n - k)$ mini-solution $r(t)$ of (1.5.1). Suppose $[t_0, t_1]$ is a compact subinterval of $[t_0, t_0 + a)$. Then there exists an $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon_0$, the k max $(n - k)$ mini-solution $r(t, \epsilon)$ of the system (1.6.3) exists on $[t_0, t_1]$, and

$$\lim_{\epsilon \to 0} r(t, \epsilon) = r(t)$$

uniformly on $[t_0, t_1]$.

1.7. Further comparison theorems

We shall be concerned, in this section, with comparison theorems for finite systems of differential inequalities. These are, naturally, extensions of some of the results in Sect. 1.4. As will be seen, minimax solutions play an essential role.

THEOREM 1.7.1. Assume that the hypotheses (i) and (iii) of Theorem 1.5.1 hold. Suppose that $[t_0, t_0 + a)$ is the largest interval in which the k max $(n - k)$ mini-solution $r(t)$ of (1.5.1) exists. Let $m \in C[[t_0, t_0 + a), R^n]$, $(t, m(t)) \in E$, $t \in [t_0, t_0 + a)$,

$$m_p(t_0) \leqslant u_{0,p}, \qquad m_q(t_0) \geqslant u_{0,q}, \tag{1.7.1}$$

and, for a fixed Dini derivative, the inequalities

$$\begin{aligned} Dm_p(t) &\leqslant g_p(t, m(t)), \\ Dm_q(t) &\geqslant g_q(t, m(t)) \end{aligned} \tag{1.7.2}$$

hold for $t \in [t_0, t_0 + a) - S$. Then,

$$m_p(t) \leqslant r_p(t), \qquad m_q(t) \geqslant r_q(t), \qquad t \in [t_0, t_0 + a). \tag{1.7.3}$$

Proof. By Lemma 1.2.2, it follows that (1.7.2) is equivalent to the inequalities

$$D_-m_p(t) \leqslant g_p(t, m(t)),$$
$$D_-m_q(t) \geqslant g_q(t, m(t)) \tag{1.7.4}$$

for $t \in (t_0, t_0 + a)$. Let $\tau \in (t_0, t_0 + a)$. Then, the existence of k max $(n - k)$ mini-solution $r(t, \epsilon)$ of (1.6.3) on $[t_0, \tau]$, for all $\epsilon > 0$ sufficiently small, satisfying

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon) \tag{1.7.5}$$

uniformly on $[t_0, \tau]$, is a consequence of Corollary 1.6.1. By Theorem 1.5.1 and the relations (1.6.3) and (1.7.4), we deduce that

$$m_p(t) < r_p(t, \epsilon),$$
$$m_q(t) > r_q(t, \epsilon) \tag{1.7.6}$$

for $t \in [t_0, \tau]$. The last inequalities, in view of (1.7.5), prove the conclusion (1.7.3).

REMARK 1.7.1. If, in Theorem 1.7.1, the inequalities (1.7.1) and (1.7.2) are reversed, the assertion (1.7.3) becomes

$$m_p(t) \geqslant \rho_p(t), \qquad m_q(t) \leqslant \rho_q(t), \qquad t \in [t_0, t_0 + a),$$

where $\rho(t)$ is the k mini $(n - k)$ max-solution of (1.5.1). The proof requires obvious changes.

The following corollary of Theorem 1:7.1 is important in later applications.

COROLLARY 1.7.1. Let condition (i) of Theorem 1.5.1 be satisfied. Suppose that g is quasi-monotone nondecreasing in u. Let $[t_0, t_0 + a)$ be the largest interval of existence of the maximal solution $r(t)$ of (1.5.1). Let $m \in C[[t_0, t_0 + a), R^n]$, $(t, m(t)) \in E$, $t \in [t_0, t_0 + a)$, and, for a fixed Dini derivative, the inequality

$$Dm(t) \leqslant g(t, m(t)) \tag{1.7.7}$$

holds for $t \in [t_0, t_0 + a) - S$. Then,

$$m(t_0) \leqslant u_0 \tag{1.7.8}$$

implies

$$m(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a). \tag{1.7.9}$$

REMARK 1.7.2. If, in Corollary 1.7.1, the inequalities (1.7.7) and (1.7.8) are reversed, then the conclusion (1.7.9) is to be replaced by

$$m(t) \geqslant \rho(t), \qquad t \in [t_0, t_0 + a),$$

where $\rho(t)$ is the minimum solution of (1.5.1). This follows from Remark 1.7.1.

The next theorem is analogous to Theorem 1.4.2 in this general framework.

THEOREM 1.7.2. Let the hypothesis of Theorem 1.7.1 hold, except that the inequalities (1.7.2) are replaced by

$$\begin{aligned} Dm_p(t) &\leqslant g_p(t, m(t)), \qquad t \in Z_p - \hat{S}_p, \\ Dm_q(t) &\geqslant g_q(t, m(t)), \qquad t \in Z_q - \hat{S}_q, \end{aligned} \tag{1.7.10}$$

where

$$Z_i = [t \in [t_0, t_0 + a) : d_i(t) < 0],$$

$$d_p(t) = r_p(t) - m_p(t), \qquad d_q(t) = m_q(t) - r_q(t),$$

and $\hat{S}_i$ is an at-most countable subset of Z_i, for each i. Then (1.7.3) remains valid.

Proof. The proof requires minor changes up to (1.7.5) of the proof of Theorem 1.7.1. Now, proceeding to prove (1.7.6) as in Theorem 1.5.1, we arrive at a t_1 and a j such that $1 \leqslant j \leqslant n$ and

$$m_j(t_1) = r_j(t_1, \epsilon). \tag{1.7.11}$$

Moreover, it is easy to obtain from Theorem 1.5.1 that

$$r_p(t) < r_p(t, \epsilon), \qquad r_q(t) > r_q(t, \epsilon) \tag{1.7.12}$$

for $t \in [t_0, \tau]$, $\tau \in (t_0, t_0 + a)$, where $r(t, \epsilon)$ is the k max $(n - k)$ mini-solution of (1.6.3) which exists on $[t_0, \tau]$, by Corollary 1.6.1. It then follows from (1.7.11) and (1.7.12) that

$$r_j(t_1) < m_j(t_1) \qquad \text{if} \quad 1 \leqslant j \leqslant k$$

or

$$r_j(t_1) > m_j(t_1) \qquad \text{if} \quad k + 1 \leqslant j \leqslant n.$$

This implies, from the definitions of Z_i and $d_i(t)$, that $t_1 \in Z_j$. Hence, the jth inequality in (1.7.10) is satisfied for such a t_1. The rest of the proof is identical with the proof of Theorem 1.5.1 in order to arrive at (1.7.6), and this is sufficient to draw the conclusion (1.7.3).

THEOREM 1.7.3. Let the hypothesis of Theorem 1.5.1 hold, except that the inequalities

$$D_{-}v_q(t) \geqslant g_q(t, v(t)), \tag{1.5.4*}$$

$$D_{-}w_p(t) \geqslant g_p(t, w(t)) \tag{1.5.5*}$$

are satisfied for $t \in (t_0, t_0 + a)$, instead of (1.5.4) and (1.5.5). Assume further, for each $\tau \in (t_0, t_0 + a)$, $t \in [t_0, \tau]$, and for each i, that g satisfies the condition

$$g_i(t, u) - g_i(t, \bar{u}) \geqslant -G(\tau + t_0 - t, u_i - \bar{u}_i), \tag{1.7.13}$$

$u_i \geqslant \bar{u}_i$, $u_j = \bar{u}_j$ $(i \neq j)$, where $G \in C[[t_0, t_0 + a) \times R, R]$, and $r(t) \equiv 0$ is the maximal solution of

$$u' = G(t, u), \qquad u(t_0) = 0.$$

Then (1.5.7) holds.

Proof. Following the proof of Theorem 1.5.1, we arrive at a $t_1 \in (t_0, t_0 + a)$ and a $j(1 \leqslant j \leqslant n)$ satisfying

$$m_j(t_1) = 0, \tag{1.7.14}$$

$$m_j(t) \geqslant 0, \qquad t_0 \leqslant t \leqslant t_1. \tag{1.7.15}$$

Let $1 \leqslant j \leqslant k$. Define $\bar{v}(t) = v(t_1 + t_0 - t)$, $\bar{w}(t) = w(t_1 + t_0 - t)$, $\bar{g}(t, u) = -g(t_1 + t_0 - t, u)$, and $d(t) = \bar{w}_j(t) - \bar{v}_j(t)$ for $t \in [t_0, t_1]$. Then, using (1.5.3) and (1.5.5*), we obtain

$$\begin{aligned} D_{-}d(t) &= D_{-}\bar{w}_j(t) - D_{-}\bar{v}_j(t) \\ &\leqslant \bar{g}_j(t, \bar{w}(t)) - \bar{g}_j(t, \bar{v}(t)). \end{aligned} \tag{1.7.16}$$

The mixed quasi-monotone property of g and the relation (1.7.15) yield that

$$\bar{g}_j(t, \bar{w}(t)) \leqslant \bar{g}_j(t, \bar{v}_1(t), \ldots, \bar{w}_j(t), \ldots, \bar{v}_n(t)).$$

This inequality implies, along with (1.7.16) and the assumption (1.7.13), the scalar differential inequality

$$D_{-}d(t) \leqslant G(t, d(t)), \qquad t \in [t_0, t_1].$$

Since $d(t_0) = 0$ because of (1.7.14), arguing as in Theorem 1.4.3, one deduces that

$$v_j(t) = w_j(t), \qquad t \in [t_0, t_1],$$

which contradicts the assumption (1.5.2) and the definition of $t_1 \in (t_0, t_0 + a)$. A repetition of the argument to the case $k + 1 \leqslant j \leqslant n$ yields a similar contradiction. This proves that the set Z is empty, and the desired result (1.5.7) follows.

COROLLARY 1.7.2. Let E be the product space $[t_0, t_0 + a) \times R^{2n}$ and $g \in C[E, R^n]$. Assume that g is quasi-monotone nondecreasing in u for each (t, v) and monotone nondecreasing in v for each (t, u). Suppose that $r(t)$ is the maximal solution of the differential system

$$u' = g(t, u, u), \qquad u(t_0) = u_0 \geqslant 0$$

existing on $[t_0, t_0 + a)$, and that $r(t) \geqslant 0$, $t \in [t_0, t_0 + a)$. Then, the maximal solution $r_1(t)$ of

$$u' = g_1(t, u), \qquad u(t_0) = u_0 \geqslant 0$$

exists on $[t_0, t_0 + a)$, where

$$g_1(t, u) = g(t, u, r(t))$$

and $r(t) = r_1(t)$, $t \in [t_0, t_0 + a)$.

COROLLARY 1.7.3. Let the assumptions of Corollary 1.7.1 hold; $m \in C[[t_0, t_0 + a), R^n]$ such that $(t, m(t), v) \in E$, $t \in [t_0, t_0 + a)$, and $m(t_0) \leqslant u_0$. Assume that for a fixed Dini derivative the inequality

$$Dm(t) \leqslant g(t, m(t), v)$$

is satisfied for $t \in [t_0, t_0 + a) - S$. Then, for all $v \leqslant r(t)$, $t \in [t_0, t_0 + a)$, we have $m(t) \leqslant r(t)$, $t \in [t_0, t_0 + a)$.

1.8. Infinite systems of differential inequalities

A classical result of Perron is that the maximal solution of a scalar differential equation can be obtained as the least upper bound of the family of functions $m(t)$ that satisfy the inequality

$$m'(t) \leqslant g(t, m(t))$$

with the same initial condition $m(t_0) = u_0$. Similar arguments hold for infinite systems of differential inequalities, provided that the maximal solution of a single equation is known. We shall first formulate an abstract

version of this method and then apply it to show the existence of minimax solutions for an infinite system of differential equations and also obtain inequalities.

Let E_1, F_1 be two partially ordered sets with the partial ordering $\leqslant$. We use the same symbol of order relation, namely, $\leqslant$, for both the sets. Assume that the following conditions hold:

$$x, y, z \in E_1, \quad x \leqslant y, \quad y \leqslant z \quad \text{imply} \quad x \leqslant z; \tag{1.8.1}$$

$$x, y \in E_1, \quad x \leqslant y, \quad y \leqslant x \quad \text{imply} \quad x = y; \tag{1.8.2}$$

$$\bar{x}, \bar{y}, \bar{z} \in F_1, \quad \bar{x} \leqslant \bar{y}, \quad \bar{y} \leqslant \bar{z} \quad \text{imply} \quad \bar{x} \leqslant \bar{z}; \tag{1.8.3}$$

$$\bar{x} \in F_1 \quad \text{implies} \quad \bar{x} \leqslant \bar{x}. \tag{1.8.4}$$

Corresponding to the sets E_1 and F_1, let us consider two partially ordered sets E_2, F_2, with the dual order relation, denoted by the symbol $\geqslant$, satisfying conditions (1.8.1*)–(1.8.4*) analogous to (1.8.1)–(1.8.4). We shall use u, v, w and $\bar{u}, \bar{v}, \bar{w}$ for elements belonging to E_2 and F_2, respectively.

Let the operators P_1, P_2 be defined on E_1, E_2, taking values in F_1, F_2, respectively. Furthermore, let the functions Q_1, Q_2 be defined on $E_1 \times E_1 \times E_2$, $E_1 \times E_2 \times E_2$, taking values in F_1, F_2, respectively.

Consider the simultaneous equations

$$\begin{aligned} P_1(x) &= Q_1(x, x, u), \\ P_2(u) &= Q_2(x, u, u). \end{aligned} \tag{1.8.5}$$

By a solution of (1.8.5), we shall mean an ordered pair (x, u), $x \in E_1$, $u \in E_2$ such that x, u satisfy Eqs. (1.8.5) simultaneously.

A solution $r = (\xi, \eta)$ of (1.8.5) is said to be a minimax solution, if for every solution (x, u) of (1.8.5) the relations

$$x \leqslant \xi, \qquad u \geqslant \eta$$

are satisfied.

The functions Q_1, Q_2 are said to possess a mixed quasi-monotone property whenever the following conditions hold: $y_1, y_2 \in E_1$, $y_1 \leqslant y_2$ imply that

$$Q_1(x, y_1, u) \leqslant Q_1(x, y_2, u), \qquad x \in E_1, \qquad u \in E_2; \tag{1.8.6}$$

and

$$Q_2(y_1, u, v) \geqslant Q_2(y_2, u, v), \qquad u, v \in E_2; \tag{1.8.7}$$

$u_1, u_2 \in E_2$, $u_1 \geqslant u_2$ imply that

$$Q_1(x, y, u_1) \leqslant Q_1(x, y, u_2), \qquad x, y \in E_1; \tag{1.8.8}$$

and

$$Q_2(y, u_1, v) \geqslant Q_2(y, u_2, v), \qquad y \in E_1, \qquad v \in E_2. \tag{1.8.9}$$

We now define the sets

$$U_1 = [x \in E_1 : P_1(x) \leqslant Q_1(x, x, u), u \in E_2]; \tag{1.8.10}$$

$$U_2 = [u \in E_2 : P_2(u) \geqslant Q_2(x, u, u), x \in E_1]. \tag{1.8.11}$$

The following theorem is concerned with the existence of the minimax solution of (1.8.5).

THEOREM 1.8.1. Let P_1, P_2, Q_1, Q_2 be as defined previously. Suppose that Q_1, Q_2 have the mixed quasi-monotone property. Assume further that there exist two functions z_1, z_2 defined on E_1, E_2 such that $z_1(E_1) \subset E_1$, $z_2(E_2) \subset E_2$, satisfying the follwing conditions:

$$\begin{aligned} P_1(z_1(x)) &= Q_1(z_1(x), x, u), \qquad u \in E_2, \\ P_2(z_2(u)) &= Q_2(x, u, z_2(u)), \qquad x \in E_1; \end{aligned} \tag{1.8.12}$$

and

$$\begin{aligned} P_1(x) &\leqslant Q_1(x, y, u), \qquad y \in E_1, \\ P_2(u) &\geqslant Q_2(y, v, u), \qquad v \in E_2 \end{aligned} \tag{1.8.13}$$

imply that $x \leqslant z_1(y)$, $u \geqslant z_2(v)$. Let the sets U_1, U_2 defined in (1.8.10) and (1.8.11) be nonempty. Then,

$$z_1(E_1) \subset U_1, \qquad z_2(E_2) \subset U_2. \tag{1.8.14}$$

Moreover, the existence of (sup U_1, inf U_2) implies the existence of (sup $z_1(U_1)$, inf $z_2(U_2)$), and vice versa. Also, sup U_1 = sup $z_1(U_1)$, inf U_2 = inf $z_2(U_2)$, and r = (sup U_1, inf U_2) is the minimax solution of (1.8.5).

Proof. Let $x \in E_1$, $u \in E_2$. Then, from (1.8.12),

$$\begin{aligned} P_1(z_1(x)) &= Q_1(z_1(x), x, u), \\ P_2(z_2(u)) &= Q_2(x, u, z_2(u)), \end{aligned} \tag{1.8.15}$$

$$x \leqslant z_1(x), \qquad u \geqslant z_2(u). \tag{1.8.16}$$

Using (1.8.4), (1.8.4*), (1.8.6), (1.8.9), and (1.8.16), we obtain, from Eqs. (1.8.15),

$$P_1(z_1(x)) \leqslant Q_1(z_1(x), z_1(x), u),$$

$$P_2(z_2(u)) \geqslant Q_2(x, z_2(u), z_2(u)),$$

which, in view of the definitions of the sets U_1, U_2, imply that

$$z_1(x) \in U_1, \qquad z_2(u) \in U_2.$$

This proves the assertion (1.8.14).

We shall show that z_1, z_2 are increasing functions. For, let $y_1 \leqslant y_2$, $v_1 \geqslant v_2$, where $y_1, y_2 \in E_1$ and $v_1, v_2 \in E_2$. Using again (1.8.6) and (1.8.9) in (1.8.12), we get

$$P_1(z_1(y_1)) \leqslant Q_1(z_1(y_1), y_2, u),$$
$$P_2(z_2(v_1)) \geqslant Q_2(y_1, v_2, z_2(v_1)),$$

and this, because of (1.8.13), proves that

$$z_1(y_1) \leqslant z_1(y_2), \qquad z_2(v_1) \geqslant z_2(v_2).$$

Suppose now that $\xi = \sup U_1$, $\eta = \inf U_2$; we have, by (1.8.14), that

$$z_1(\xi) \leqslant \xi, \qquad z_2(\eta) \geqslant \eta. \tag{1.8.17}$$

On the other hand, since $x \leqslant \xi$, $u \geqslant \eta$ for any $x \in U_1$, $u \in U_2$, the mixed quasi-monotone property of Q_1, Q_2, together with the definitions of U_1, U_2, yields

$$P_1(x) \leqslant Q_1(x, \xi, \eta),$$
$$P_2(u) \geqslant Q_2(\xi, \eta, u).$$

It then follows from (1.8.13) that

$$x \leqslant z_1(\xi), \qquad u \geqslant z_2(\eta).$$

Consequently, $\xi = z_1(\xi)$, $\eta = z_2(\eta)$ from (1.8.17). It is evident from (1.8.12) that $r = (\xi, \eta)$ is a solution of (1.8.5).

Let $y \leqslant y^*$, $v \geqslant v^*$ for $y \in z_1(U_1)$, $v \in z_2(U_2)$. Then

$$z_1(\xi) \leqslant y^*, \qquad z_2(\eta) \geqslant v^*.$$

The monotonic nature of z_1, z_2 yields

$$y = z_1(x) \leqslant z_1(\xi) \qquad \text{for all} \quad y \in z_1(U_1),$$
$$v = z_2(u) \geqslant z_2(\eta) \qquad \text{for all} \quad v \in z_2(U_2).$$

It therefore implies that $\xi = \sup z_1(U_1)$, $\eta = \inf z_2(U_2)$. The fact that $r = (\xi, \eta)$ is the minimax solution of (1.8.5) follows from the definitions of U_1, U_2.

If $\xi = \sup z_1(U_1)$, $\eta = \inf z_2(U_2)$, using similar arguments, one can prove that $\xi = \sup U_1$, $\eta = \inf U_2$, respectively, and that $r = (\xi, \eta)$ is the minimax solution of (1.8.5). This completes the proof of the theorem.

Consider the system

$$\begin{aligned} u_i' &= f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q), \qquad && u_i(t_0) = u_i^0, \\ & && 1 \leqslant i \leqslant p, \\ v_j' &= g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q), && v_j(t_0) = v_j^0, \\ & && 1 \leqslant j \leqslant q, \end{aligned} \tag{1.8.18}$$

where p, q are arbitrary (may be infinite). The existence of minimax solutions for finite systems of differential equations follows when p, q are both finite. The minimax solution for infinite systems is covered by the other choices of p and q (either p or q or both may be infinite). In case p, q are both infinite, the functions f_i and g_j in (1.8.18) are to be interpreted as $f_i(t, u_1, u_2, \ldots; v_1, v_2, \ldots)$ and $g_j(t, u_1, u_2, \ldots; v_1, v_2, \ldots)$, respectively.

Let the functions f_i and g_j satisfy the following assumptions:

(i) $f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ and $g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ are defined for $[t_0, t_0 + a]$ and arbitrary $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$,

(ii) There exists constants M_i and N_j such that

$$|f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)| \leqslant M_i,$$

$$|g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)| \leqslant N_j$$

for $t \in [t_0, t_0 + a]$, $1 \leqslant i \leqslant p$, and $1 \leqslant j \leqslant q$.

(iii) The functions f_i and g_j are continuous in the sense that, if $t^\alpha \to t$, $u_i^\alpha \to u_i$, $v_j^\alpha \to v_j$ $(1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q)$, then

$$f_i(t^\alpha, u_1^\alpha, u_2^\alpha, \ldots, u_p^\alpha; v_1^\alpha, v_2^\alpha, \ldots, v_q^\alpha) \to f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q),$$

$$g_j(t^\alpha, u_1^\alpha, u_2^\alpha, \ldots, u_p^\alpha; v_1^\alpha, v_2^\alpha, \ldots, v_q^\alpha) \to g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q).$$

(iv) The functions f_i and g_j possess mixed quasi-monotone property, i.e., for each $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$,

(a_1) $f_i(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ is monotonic nondecreasing in $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_p$ and monotonic nonincreasing in $v_1, v_2, \ldots, v_q$;

(a$_2$) $g_j(t, u_1, u_2, \ldots, u_p; v_1, v_2, \ldots, v_q)$ is monotonic nonincreasing in $u_1, u_2, \ldots, u_p$ and monotonic nondecreasing in $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_q$.

Now we have the following:

THEOREM 1.8.2. Assume that the functions f_i and g_j satisfy conditions (i)–(iv). Then there exists a minimax solution $(u_i^*(t), v_j^*(t))$ of (1.8.18) on $[t_0, t_0 + a]$. Furthermore, if $m_i(t)$, $n_j(t)$ are continuous functions defined on $[t_0, t_0 + a]$, satisfying the inequalities

$$m_i(t_0) \leqslant u_i^0, \qquad n_j(t_0) \geqslant v_j^0,$$

$$D^+m_i(t) \leqslant f_i(t, m_1(t), m_2(t), \ldots, m_p(t); n_1(t), n_2(t), \ldots, n_q(t)),$$

$$D^+n_j(t) \geqslant g_j(t, m_1(t), m_2(t), \ldots, m_p(t); n_1(t), n_2(t), \ldots, n_q(t)),$$

and

$$m_i(t) \leqslant u_i^*(t), \qquad n_j(t) \geqslant v_j^*(t), \qquad t \in [t_0, t_0 + a].$$

Proof. Let $\{\phi_i(t)\}$, $\{\psi_j(t)\}$ be two sequences of continuous functions on $[t_0, t_0 + a]$ such that, for $t \in [t_0, t_0 + a]$,

$$\phi_i(t) \leqslant u_i^0 + M_i(t - t_0), \qquad i = 1, 2, \ldots, p,$$

$$\psi_j(t) \leqslant v_j^0 + N_j(t - t_0), \qquad j = 1, 2, \ldots, q.$$

Denote

$$\{\phi_i(t)\} = (\phi_1(t), \phi_2(t), \ldots, \phi_p(t)) = y,$$

$$\{\psi_j(t)\} = (\psi_1(t), \psi_2(t), \ldots, \psi_q(t)) = v.$$

Let E_1, E_2 stand for systems of sequences $y = \{\phi_i(t)\}$, $v = \{\psi_j(t)\}$. If $y_1 = \{\phi_i^{(1)}(t)\}$, $y_2 = \{\phi_i^{(2)}(t)\}$ are two sequences such that, if $y_1, y_2 \in E_1$, then $y_1 \leqslant y_2$ implies $\phi_i^{(1)}(t) \leqslant \phi_i^{(2)}(t)$ on $[t_0, t_0 + a]$ and for each i. Similarly for $v_1, v_2 \in E_2$, $v_1 \geqslant v_2$ means that $\psi_j^{(1)}(t) \geqslant \psi_j^{(2)}(t)$ on $[t_0, t_0 + a]$ and for each j.

Let F_1, F_2 stand for the systems of sequences of continuous functions on $[t_0, t_0 + a]$ which take values in the real extended line, the order relations in F_1, F_2 being the same as those in E_1, E_2, respectively. It is easy to verify that conditions (1.8.1)–(1.8.4) and (1.8.1*)–(1.8.4*) are satisfied. Let us now define the operators P_1, P_2 and the functions Q_1, Q_2 as follows:

$$P_1(y) = \{D^+\phi_i(t)\}, \qquad P_2(v) = \{D^+\psi_j(t)\},$$

$$Q_1(x, y, v) = f_i(t, \phi_1(t), \ldots, \phi_{i-1}(t), x_i(t), \phi_{i+1}(t), \ldots, \phi_p(t); \psi_1(t), \psi_2(t), \ldots, \psi_q(t)),$$

$$Q_2(y, v, u) = g_j(t, \phi_1(t), \phi_2(t), \ldots, \phi_p(t); \psi_1(t), \ldots, \psi_{j-1}(t), u_j(t), \psi_{j+1}(t), \ldots, \psi_q(t)).$$

Clearly the functions Q_1, Q_2 satisfy the mixed quasi-monotone property. For any pair of sequences $\{\phi_i(t)\}$, $\{\psi_j(t)\}$, define

$$\bar{f}_i(t, r) = f_i(t, \phi_1(t),..., \phi_{i-1}(t), r_i, \phi_{i+1}(t),..., \phi_p(t); \psi_1(t), \psi_2(t),..., \psi_q(t));$$

$$\bar{g}_j(t, \rho) = g_j(t, \phi_1(t),..., \phi_p(t); \psi_1(t),..., \psi_{j-1}(t), \rho_j, \psi_{j+1}(t),..., \psi_q(t)).$$

Let, for each i, $r_i(t)$ be the maximal solution of

$$r' = \bar{f}_i(t, r), \qquad r_i(t_0) = u_i^0,$$

and, for each j, $\rho_j(t)$ be the minimal solution of

$$\rho' = \bar{g}_j(t, \rho), \qquad \rho_j(t_0) = v_j^0.$$

Since the functions f_i and g_j satisfy (ii), the existence of $r_i(t)$ and $\rho_j(t)$ on $[t_0, t_0 + a]$ is ensured. Let us now define the functions z_1, z_2 by

$$z_1(\{\phi_i(t)\}) = \{r_i(t)\}, \qquad z_2(\{\psi_j(t)\}) = \{\rho_j(t)\}.$$

From Theorem 1.4.1, it follows that the functions z_1, z_2 satisfy (1.8.13). Moreover, the sets U_1, U_2 are nonempty, since $u_i^0 - M_i(t - t_0) \in U_1$ and $v_j^0 + N_j(t - t_0) \in U_2$, because of (ii). Furthermore,

$$| r_i'(t)| \leq M_i, \qquad | \rho_j'(t)| \leq N_j.$$

Therefore, the family of functions $\{r_i(t)\}$, $\{\rho_j(t)\}$ are equicontinuous and uniformly bounded. This proves that

$$\sup z_1(U_1) = \{\sup r_i(t)\},$$

$$\inf z_2(U_2) = \{\inf \rho_j(t)\}$$

are continuous on $[t_0, t_0 + a]$. The assertion of Theorem 1.8.2 now follows from Theorem 1.8.1.

1.9. Integral inequalities reducible to differential inequalities

We shall consider, in this section, only those integral inequalities that are reducible to differential inequalities. We begin with one of the simplest and most useful integral inequalities.

THEOREM 1.9.1. Let $m, v \in C[[t_0, t_0 + a), R_+]$, where R_+ denotes the nonnegative real line. Suppose further that, for some nonnegative constant C, we have

$$m(t) \leq C + \int_{t_0}^{t} v(s)m(s)\, ds, \qquad t \in [t_0, t_0 + a). \tag{1.9.1}$$

Then

$$m(t) \leqslant C \exp \int_{t_0}^{t} v(s)\, ds, \qquad t \in [t_0, t_0 + a). \tag{1.9.2}$$

Proof. If $C > 0$, it follows from (1.9.1) that

$$\frac{m(t)\, v(t)}{C + \int_{t_0}^{t} v(s)\, m(s)\, ds} \leqslant v(t),$$

which, by integration, yields

$$\log \left\{ C + \int_{t_0}^{t} v(s)\, m(s)\, ds \right\} - \log C \leqslant \int_{t_0}^{t} v(s)\, ds.$$

This inequality, together with (1.9.1), gives (1.9.2).

If $C = 0$, then (1.9.1) holds for every constant $C_1 > 0$, and therefore the previous argument gives (1.9.2) with $C = C_1$. Letting $C_1 \to 0$ implies $m(t) \equiv 0$. This proves the theorem.

COROLLARY 1.9.1. Let $m, v \in C[[t_0, t_0 + a), R_+]$, $n \in C[[t_0, t_0 + a), R]$, and satisfy the inequality

$$m(t) \leqslant n(t) + \int_{t_0}^{t} v(s)\, m(s)\, ds, \qquad t \in [t_0, t_0 + a).$$

Then we have

$$m(t) \leqslant n(t) + \int_{t_0}^{t} v(s)\, n(s) \left(\exp \int_{s}^{t} v(\xi)\, d\xi \right) ds, \qquad t \in [t_0, t_0 + a).$$

If, in addition, the derivative $n'(t)$ exists for $t \in [t_0, t_0 + a)$, then

$$m(t) \leqslant n(t_0) \left(\exp \int_{t_0}^{t} v(s)\, ds \right) + \int_{t_0}^{t} \exp \left(\int_{s}^{t} v(\xi)\, d\xi \right) n'(s)\, ds.$$

A generalization of Theorem 1.9.1 is the following analog of Theorem 1.4.1 which, however, requires the monotony of g with respect to u.

THEOREM 1.9.2. Let E be an open (t, u)-set in R^2 and $g \in C[E, R]$. Suppose that $g(t, u)$ is monotonic nondecreasing in u for each t. Let $m \in C[[t_0, t_0 + a), R]$, $(t, m(t)) \in E$, $t \in [t_0, t_0 + a)$, $m(t_0) \leqslant u_0$, and

$$m(t) \leqslant m(t_0) + \int_{t_0}^{t} g(s, m(s))\, ds, \qquad t \in [t_0, t_0 + a). \tag{1.9.3}$$

Then

$$m(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a), \tag{1.9.4}$$

where $r(t)$ is the maximal solution of (1.2.1) existing on J.

Proof. Define

$$v(t) = m(t_0) + \int_{t_0}^{t} g(s, m(s))\, ds,$$

so that

$$m(t) \leqslant v(t) \tag{1.9.5}$$

and

$$v'(t) = g(t, m(t)).$$

Since g is monotonic in u, using (1.9.5), we obtain the differential inequality

$$v'(t) \leqslant g(t, v(t)), \qquad t \in [t_0, t_0 + a).$$

From an application of Theorem 1.4.1, we deduce that

$$v(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a).$$

The assertion (1.9.4) is now immediate because of (1.9.5).

REMARK 1.9.1. Notice that one could prove Theorem 1.9.1 using an argument similar to that of Theorem 1.9.2, although we have given the classical proof.

COROLLARY 1.9.2. Let $m, v \in C[[t_0, t_0 + a), R_+]$, $g \in C[R_+, R_+]$, $g(u)$ monotone in u and $g(0) = 0$. Assume that

$$m(t) \leqslant m_0 + \int_{t_0}^{t} v(s)\, g(m(s))\, ds, \qquad t \in [t_0, t_0 + a).$$

Then

$$m(t) \leqslant w^{-1}[w(m_0) + \int_{t_0}^{t} v(s)\, ds], \qquad t_0 \leqslant t \leqslant t_1,$$

where

$$w(u) = \int_{u_0}^{u} d\tau/g(\tau), \qquad u_0 > 0;$$

$w^{-1}(u)$ is the inverse function of $w(u)$, and $(t_0, t_1) \subset [t_0, t_0 + a)$ such that $w(m_0) + \int_{t_0}^{t} v(s)\, ds$ is in the domain of definition of $w^{-1}(u)$.

COROLLARY 1.9.3. Let $m, p \in C[[t_0, t_0 + a), R]$ and

$$m(t) \leqslant m(t_0) + \int_{t_0}^{t} [K\, m(s) + p(s)]\, ds, \qquad K > 0.$$

Then

$$m(t) \leqslant m(t_0) \exp[K(t - t_0)] + \int_{t_0}^{t} p(s) \exp[K(t - s)]\, ds, \qquad t \in [t_0, t_0 + a).$$

COROLLARY 1.9.4. Let the assumptions of Theorem 1.9.2 be satisfied except that the integral inequality (1.9.3) be replaced by

$$m(t) \leqslant n(t) + \int_{t_0}^{t} g(s, m(s))\, ds, \qquad t \in [t_0, t_0 + a),$$

where $n \in C[[t_0, t_0 + a), R]$. Then (1.9.4) takes the form

$$m(t) \leqslant n(t) + r(t), \qquad t \in [t_0, t_0 + a),$$

where $r(t)$ is the maximal solution of

$$u' = g(t, n(t) + u), \qquad u(t_0) = 0$$

existing on $[t_0, t_0 + a)$.

It is easy to extend Theorem 1.9.2 to finite systems of integral inequalities. Actually, we prove such a result in a more general form.

THEOREM 1.9.3. Let assumption (i) of Theorem 1.5.1 hold, and suppose that $g(t, u)$ has the mixed monotone property in u. Let $m \in C[[t_0, t_0 + a), R^n]$, $(t, m(t)) \in E$, $t \in [t_0, t_0 + a)$, and the inequalities

$$m_p(t_0) \leqslant u_{0,p}, \qquad m_q(t_0) \geqslant u_{0,q},$$

$$m_p(t) \leqslant m_p(t_0) + \int_{t_0}^{t} g_p(s, m(s))\, ds,$$

$$m_q(t) \geqslant m_q(t_0) + \int_{t_0}^{t} g_q(s, m(s))\, ds$$

hold for $t \in [t_0, t_0 + a)$. Then

$$m_p(t) \leqslant r_p(t), \qquad m_q(t) \geqslant r_q(t) \tag{1.9.6}$$

for $t \in [t_0, t_0 + a)$, where $r(t)$ is the k max $(n - k)$ mini-solution of (1.5.1).

Proof. Let the vector function $v(t)$ be equal to

$$m(t_0) + \int_{t_0}^{t} g(s, m(s))\, ds,$$

so that

$$m_p(t) \leqslant v_p(t), \qquad m_q(t) \geqslant v_q(t) \tag{1.9.7}$$

and

$$v'(t) = g(t, v(t)).$$

The mixed monotonic character of g in u shows, in view of the inequalities (1.9.7), that

$$\begin{aligned} v_p'(t) &\leqslant g_p(t, v(t)), \\ v_q'(t) &\geqslant g_q(t, v(t)), \qquad t \in [t_0, t_0 + a). \end{aligned}$$

Theorem 1.7.1 is now applicable, and we get

$$v_p(t) \leqslant r_p(t), \qquad v_q(t) \geqslant r_q(t), \qquad t \in [t_0, t_0 + a).$$

The inequalities (1.9.6) result from (1.9.7), and the theorem is proved.

1.10. Differential inequalities in the sense of Caratheodory

Let the function $g(t, u)$ be defined on an open (t, u)-set $E \subset R^2$, taking values in R. $g(t, u)$ is said to satisfy the Caratheodory condition if (i) $g(t, u)$ is continuous in u for each fixed t and Lebesgue measurable in t for each fixed u; and (ii) $M(t)$ is a summable function on $[t_0, t_0 + a]$ and

$$| g(t, u)| \leqslant M(t), \qquad (t, u) \in E.$$

By a solution $u(t)$ of the differential equation with an initial condition

$$u' = g(t, u), \qquad u(t_0) = u_0, \tag{1.10.1}$$

we mean an absolutely continuous function $u(t)$ satisfying (1.10.1) almost everywhere on $[t_0, t_0 + a]$.

By the classical theorem of Caratheodory, there exists a solution of (1.10.1) under the foregoing conditions. Moreover, existence of maximal and minimal solutions and the problem of extension of solutions can be shown in just a similar way as before.

The following theorem on differential inequalities of Caratheodory type is of interest.

THEOREM 1.10.1 Let (i) the function $g(t, u)$ be defined on an open (t, u)-set $E \subset R^2$, taking values in R and satisfying the Caratheodory's condition; and (ii) $r(t)$ be the maximal solution of (1.10.1) existing on $[t_0, t_0 + a]$. Assume that $m \in C[[t_0, t_0 + a], R]$ and is of bounded variation on $[t_0, t_0 + a]$ such that its singular part is a nonincreasing function. Suppose further that

$$m'(t) \leqslant g(t, m(t)) \tag{1.10.2}$$

almost everywhere on $[t_0, t_0 + a]$. Then

$$m(t_0) \leqslant u_0 \tag{1.10.3}$$

implies

$$m(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a]. \tag{1.10.4}$$

Proof. Define the function

$$f(t, u) = \begin{cases} g(t, u) & \text{if} \quad m(t) \leqslant u, \\ g(t, m(t)) & \text{if} \quad u \leqslant m(t), \end{cases}$$

which satisfies the Caratheodory's condition. Let $r_1(t)$ denote the maximal solution of

$$u' = f(t, u), \qquad u(t_0) = u_0. \tag{1.10.5}$$

We claim that

$$m(t) \leqslant r_1(t), \qquad t \in [t_0, t_0 + a]. \tag{1.10.6}$$

If this were not true, let, without any loss of generality, (t_1, t_2) be an open interval such that

$$m(t_1) = r_1(t_1) \tag{1.10.7}$$

and

$$m(t_1 + h) > r_1(t_1 + h), \tag{1.10.8}$$

$h > 0$ sufficiently small. For $t \in (t_1, t_2)$, we obtain from (1.10.2) and (1.10.5) the inequality

$$m'(t) - r_1'(t) \leqslant g(t, m(t)) - f(t, r_1(t)). \tag{1.10.9}$$

The definition of $f(t, u)$, together with (1.10.7), (1.10.8), and (1.10.9), gives

$$m'(t) - r_1'(t) \leqslant 0,$$

which in its turn implies

$$\int_{t_1}^{t_1+h} m'(s)\, ds \leqslant r_1(t_1 + h) - r_1(t_1). \tag{1.10.10}$$

Since $m(t)$ is of bounded variation, we have

$$m(t_1 + h) = m(t_1) + \int_{t_1}^{t_1+h} m'(s)\, ds + \beta(t_1 + h), \qquad (1.10.11)$$

where $\beta(t)$ is the singular part of $m(t)$. The relations (1.10.7), (1.10.10), and (1.10.11) lead to

$$m(t_1 + h) - \beta(t_1 + h) \leqslant r_1(t_1 + h).$$

As $\beta(t_1) = 0$ and $\beta(t)$ is nonincreasing by assumption, $-\beta(t_1 + h) \geqslant 0$, and consequently

$$m(t_1 + h) \leqslant r_1(t_1 + h),$$

which contradicts (1.10.8). This proves (1.10.6).

The definition of f now shows, because of (1.10.6), that $r_1(t)$ is a solution of (1.10.1), and therefore

$$r_1(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a].$$

This completes the proof.

THEOREM 1.10.2. Let assumptions (i) and (ii) of Theorem 1.10.1 be satisfied. Assume that $m(t)$ is absolutely continuous for $t \in [t_0, t_0 + a]$ and satisfies (1.10.2) almost everywhere on $[t_0, t_0 + a]$. Then (1.10.3) implies (1.10.4).

If $m(t)$ is absolutely continuous on $[t_0, t_0 + a]$, the singular part $\beta(t)$ of $m(t)$ is identically zero. This remark shows that Theorem 1.10.2 is a consequence of Theorem 1.10.1.

THEOREM 1.10.3. Let assumption (i) of Theorem 1.10.1 hold. Assume that $g(t, u) \geqslant 0$; $m \in C[[t_0, t_0 + a], R]$ and satisfies, for small $h > 0$,

$$| m(t + h) - m(t)| \leqslant \int_t^{t+h} g(s, m(s))\, ds, \qquad t, t + h \in [t_0, t_0 + a]. \qquad (1.10.12)$$

Then (1.10.3) implies (1.10.4).

Proof. It follows from the inequality (1.10.12) that $m(t)$ is absolutely continuous over any interval in $[t_0, t_0 + a]$. Consequently, $m'(t)$ exists almost everywhere on $[t_0, t_0 + a]$. Moreover, (1.10.12) implies that the derivative $m'(t)$ satisfies the relation

$$| m'(t)| \leqslant g(t, m(t))$$

almost everywhere on $[t_0, t_0 + a]$. The assertion (1.10.4) now follows from Theorem 1.10.2.

1.11. Notes

Theorems 1.1.2 and 1.1.3 are taken from Hartman [5]. Theorem 1.1.2 is due to Peano [2], and the proof uses a device of Tonelli [1]. For the type of results on differential inequalities in Sect. 1.2, see Babkin [1], Caferio [1, 2] and Chaplygin [1].

Maximal and minimal solutions are considered by Peano [1]. Also see Kamke [1] and Perron [1]. Lemma 1.3.2 is due to Wazewski [9]. Differential inequalities of the type (1.4.1) occur in the work of Peano [1] and of Perron [1]. Theorem 1.4.3 is due to Walter [3]. The result of Theorem 1.4.4 is due to Wazewski (see Szarski [1]). The proof in the text does not require $g(t, u) \geqslant 0$. Theorem 1.4.5 is new.

The results of Sects. 1.5, 1.6, and 1.7 have been adopted from the work of Burton and Whyburn [1], who introduced the notion of minimax solutions. See also Kamke [2] and Wazewski [3, 4, 6] for results on extremal solutions for systems and corresponding theorems on differential inequalities.

Section 1.8 contains the work of Lakshmikantham and Leela [4]. For allied results, see Mlak and Olech [1] and Mlak [2].

For Theorem 1.9.1, see Bellman [3], Giuliano [1], Gronwall [1], and Peano [1]. Corollary 1.9.2 is due to Bihari [1]. Also, see Lakshmikantham [9] and Langenhop [1]. Theorem 1.9.2 is due to Viswanatham [2, 4]. See also Baiada [1] and Cafiero [3]. For a generalization of this result to systems, see Lakshmikantham [2] and Opial [1]. Theorem 1.9.3 is new. Theorem 1.10.1 is taken from Olech and Opial [1].

The result of Theorem 1.10.3 is due to Lakshmikantham [1]. See also Olech and Opial [1]. For Caratheodory's existence theorem, see McShane [1].

New proofs of Theorems 1.1.2, 1.3.1, and 1.4.1 are given by Corduneanu [17]. For differential inequalities with the limiting initial conditions, see Mamedov [1]. For collateral reading on differential and integral inequalities, see Szarski [1] and Walter [3].

Chapter 2

2.0. Introduction

The most important techniques in the theory of differential equations involve a systematic use of the theory of differential inequalities, or what may be termed as the integration of differential inequalities. We present, in this chapter, a number of varied results depending essentially on this approach. Many of our theorems and their proofs will involve estimates with respect to some convenient norm. The choice of a norm as a medium for our arguments is a natural one, although it is seldom the best choice. A better candidate is, of course, the so-called Lyapunov function, which is more flexible. An approach based on Lyapunov functions will be postponed to later chapters.

2.1. Global existence

We shall use the Tychonoff's fixed point theorem for locally convex linear spaces to prove the global existence of solutions of the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{2.1.1}$$

where $f \in C[J \times R^n, R^n]$, J being the half-line $[0, \infty)$. Let us state the fixed point theorem of Tychonoff in the following form.

THEOREM 2.1.1. Let B be a complete, locally convex, linear space and B_0 a closed convex subset of B. Let the mapping $T : B \to B$ be continuous and $T(B_0) \subset B_0$. If $\overline{T(B_0)}$ is compact, then T has a fixed point in B_0.

The main result of this section runs as follows.

THEOREM 2.1.2. Let $f \in C[J \times R^n, R^n]$, and, for $(t, x) \in J \times R^n$,

$$\|f(t, x)\| \leq g(t, \|x\|), \tag{2.1.2}$$

where $g \in C[J \times R_+, R_+]$ and the function $g(t, u)$ is monotonic nondecreasing in u for each $t \in J$. Assume that, for every $u_0 > 0$, the scalar differential equation

$$u' = g(t, u), \qquad u(t_0) = u_0, \qquad t_0 \geq 0 \tag{2.1.3}$$

has a solution $u(t) = u(t, t_0, u_0)$ existing for $t \geq t_0$. Then, for every $x_0 \in R^n$ such that $\|x_0\| \leq u_0$, there exists a solution $x(t) = x(t, t_0, x_0)$ of (2.1.1) for $t \geq t_0$, satisfying

$$\|x(t)\| \leq u(t), \qquad t \geq t_0.$$

Proof. To apply Theorem 2.1.1, let us consider the real vector space B of all continuous functions from $[t_0, \infty)$ into R^n, the topology on B being that induced by the family of pseudo-norms $\{p_n(x)\}_{n=1}^{\infty}$, where for $x \in B$,

$$p_n(x) = \sup_{t_0 \leq t \leq n} \|x(t)\|.$$

A fundamental system of neighborhoods is then given by

$$\{V_n\}_{n=1}^{\infty}, \qquad \text{where} \quad V_n = [x \in B : p_n(x) \leq 1].$$

Under this topology, B becomes a complete, locally convex, linear space. Let us now define a subset B_0 of B as follows:

$$B_0 = [x \in B : \|x(t)\| \leq u(t), t \geq t_0], \tag{2.1.4}$$

where $u(t)$ is a solution of (2.1.3) existing for $t \geq t_0$. It is clear that, in the topology of B, the set B_0 is closed, convex, and bounded. Consider the integral operator defined by

$$T(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \tag{2.1.5}$$

whose fixed point corresponds to a solution of the system (2.1.1). Evidently, the operator T is compact in the topology of B, and therefore $\overline{T(B_0)}$ is compact in view of the boundedness of B_0. To prove the theorem, it remains to be shown that $T(B_0) \subset B_0$. To this end, we observe that for any $x \in B_0$,

$$\|T(x)(t)\| \leq \|x_0\| + \int_{t_0}^{t} g(s, \|x(s)\|)\, ds, \tag{2.1.6}$$

because of (2.1.5) and (2.1.2). Using the monotonic character of $g(t, u)$ in u, the definition of B_0, and the fact that $u(t)$ is a solution of (2.1.3) such that $\| x_0 \| \leqslant u_0$, it follows from (2.1.6) that

$$\| T(x)(t) \| \leqslant u(t).$$

This implies $T(B_0) \subset B_0$ and completes the proof.

COROLLARY 2.1.1. Let $f \in C[J \times R^n, R^n]$ and

$$\| f(t, x) \| \leqslant \lambda(t) g(\| x \|) \tag{2.1.7}$$

for $(t, x) \in J \times R^n$, where $\lambda(t) \geqslant 0$ is continuous for $t \in J$; $g(u) \geqslant 0$ is continuous for $u \geqslant 0$, $g(0) = 0$, $g(u) > 0$, $u > 0$, and $g(u)$ is nondecreasing in u. If

$$\int_{u_0}^{\infty} du/g(u) = \infty \tag{2.1.8}$$

for $u_0 > 0$, then, for every $x_0 \in R^n$, there exists a solution of (2.1.1) for $t \geqslant t_0$.

Proof. The result follows from Theorem 2.1.2, if we show that the differential equation (2.1.3) has a solution existing for $t \geqslant t_0$. Clearly, the equation

$$u' = \lambda(t) g(u), \qquad u(t_0) = u_0 > 0 \tag{2.1.9}$$

may be solved. For, if we write

$$G(u) = \int_{u_0}^{u} du/g(u) = \int_{t_0}^{t} \lambda(s)\, ds,$$

it is easily seen that the function $G(u)$ is strictly increasing in u, and so the inverse function exists. In view of the assumptions concerning g, the domain of the inverse function is $[0, \infty)$, and therefore the solution $u(t)$ of (2.1.9) is defined for $t \geqslant t_0$.

REMARK 2.1.1. It is possible to obtain, from Theorem 2.1.2, some qualitative information of solutions of (2.1.1), knowing the behavior of solutions of the related scalar differential equation (2.1.3). The result that follows illustrates this fact.

THEOREM 2.1.3. Consider the system

$$x' = A(t)x + f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{2.1.10}$$

where $A(t)$ is a continuous, $n \times n$ matrix for $t \in J$ and $f \in C[J \times R^n, R^n]$. Let $U(t)$ denote the principal matrix solution of the linear system

$$x' = A(t)x$$

for $t \geqslant t_0$, satisfying the estimates

$$\| U(t)\| \leqslant Ke^{-\sigma t},$$

$$\| U(t)U^{-1}(t)\| \leqslant Ke^{-\sigma(t-s)}, \qquad K > 0, \quad \sigma > 0, \quad t \geqslant s.$$

Assume that, for $(t, x) \in J \times R^n$,

$$\| f(t, x)\| \leqslant g(t, \| x \|),$$

where $g \in C[J \times R_+, R_+]$, $g(t, u)$ is monotone nondecreasing in u for each fixed t, and the scalar differential equation

$$u' = -\sigma u + Kg(t, u), \qquad u(t_0) = u_0 > 0$$

has a solution $u(t)$ existing for $t \geqslant t_0$ such that

$$\lim_{t \to \infty} u(t) = 0.$$

Then, there exists a solution $x(t)$ of (2.1.10) defined for $t \geqslant t_0$ such that

$$\lim_{t \to \infty} x(t) = 0.$$

2.2. Uniqueness

A simple criterion that implies the uniqueness of solutions is the following Perron's condition.

THEOREM 2.2.1. Assume that (i) the function $g(t, u)$ is continuous and nonnegative for $t_0 \leqslant t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$, and, for every t_1, $t_0 < t_1 < t_0 + a$, $u(t) \equiv 0$ is the only differentiable function on $t_0 \leqslant t < t_1$, which satisfies

$$u' = g(t, u), \qquad u(t_0) = 0 \tag{2.2.1}$$

for $t_0 \leqslant t < t_1$; (ii) $f \in C[R_0, R^n]$, where

$$R_0 : t_0 \leqslant t \leqslant t_0 + a, \qquad \| x - x_0 \| \leqslant b,$$

and for $(t, x), (t, y) \in R_0$,

$$\|f(t, x) - f(t, y)\| \leqslant g(t, \|x - y\|). \tag{2.2.2}$$

Then, the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{2.2.3}$$

has at most one solution on $t_0 \leqslant t \leqslant t_0 + a$.

Proof. Suppose that there are two solutions $x_1(t)$ and $x_2(t)$ of the system (2.2.3) on $t_0 \leqslant t \leqslant t_0 + a$. Define $m(t) = \| x_1(t) - x_2(t) \|$. Then,

$$\begin{aligned} D^+m(t) &\leqslant \| x_1'(t) - x_2'(t)\| = \|f(t, x_1(t)) - f(t, x_2(t))\| \\ &\leqslant g(t, m(t)), \end{aligned}$$

using (2.2.2). Also, $m(t_0) = 0$. For any t_1 such that $t_0 < t_1 < t_0 + a$, we obtain from Theorem 1.4.1 the inequality

$$m(t) \leqslant r(t), \qquad t_0 \leqslant t < t_1,$$

where $r(t)$ is the maximal solution of (2.2.1). The assumption (*i*) now assures that $m(t) \equiv 0$ on $t_0 \leqslant t < t_1$, proving the theorem.

COROLLARY 2.2.1. The function $g(t, u) = Ku$, $K > 0$, is admissible in Theorem 2.2.1.

It is an easy exercise to verify that $g(t, u)$ satisfies assumption (i) of Theorem 2.2.1. In this case, the condition (2.2.2) just reduces to the well-known Lipschitz condition.

Although Corollary 2.2.1 is a direct consequence of Theorem 2.2.1, we give below a proof that is instructive.

Proof of Corollary 2.2.1. Let $m(t)$ be the same function defined previously, and let $m(t) = n(t)e^{Lt}$, where $L > K$ is a constant. It is enough to show that $n(t) \equiv 0$ on $t_0 \leqslant t \leqslant t_0 + a$. Suppose, on the contrary, that

$$\max_{t_0 \leqslant t \leqslant t_0+a} n(t) > 0,$$

and that the maximum occurs at $t = \sigma$. We have, at σ, $n'(\sigma) = 0$. On the other hand, using (2.2.2), we obtain

$$\begin{aligned} n'(\sigma)e^{L\sigma} + Ln(\sigma)e^{L\sigma} &= m'(\sigma) \\ &\leqslant \|f(\sigma, x_1(\sigma)) - f(\sigma, x_2(\sigma))\| \\ &\leqslant Kn(\sigma)e^{L\sigma}. \end{aligned}$$

This implies, because of the choice $K < L$, that $n'(\sigma) < 0$, contradicting $n'(\sigma) = 0$. Thus, $n(t) = 0$ on $t_0 \leqslant t \leqslant t_0 + a$.

The next result is known as Kamke's uniqueness theorem, which is, evidently, more general than that of Perron and is sufficient for many practical cases, since it includes as special cases many known criteria.

THEOREM 2.2.2. Assume that (i) the function $g(t, u)$ is continuous and nonnegative for $t_0 < t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$, and, for every t_1, $t_0 < t_1 < t_0 + a$, $u(t) = 0$ is the only function differentiable on $t_0 < t < t_1$ and continuous on $t_0 \leqslant t < t_1$, for which

$$u'_+(t_0) = \lim_{t \to t_0^+} \frac{u(t) - u(t_0)}{t - t_0} \qquad \text{exists,}$$

$$u'(t) = g(t, u(t)), \qquad t_0 < t < t_1, \tag{2.2.4}$$

$$u(t_0) = u'_+(t_0) = 0; \tag{2.2.5}$$

(ii) the hypothesis (ii) of Theorem 2.2.1 is satisfied except that the condition (2.2.2) holds for $(t, x), (t, y) \in R_0$, $t \neq t_0$.

Then, the conclusion of Theorem 2.2.1 is valid.

We shall first prove the following:

THEOREM 2.2.3. Let the function $g(t, u)$ verify hypothesis (i) of Theorem 2.2.2. Assume that the function $g_1(t, u)$ is continuous and nonnegative for $t_0 \leqslant t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$, $g_1(t, 0) = 0$, and

$$g_1(t, u) \leqslant g(t, u), \qquad t \neq t_0. \tag{2.2.6}$$

Then, for every t_1, $t_0 < t_1 < t_0 + a$, $u(t) = 0$ is the only differentiable function on $t_0 \leqslant t < t_1$, which satisfies

$$u' = g_1(t, u), \qquad u(t_0) = 0 \tag{2.2.7}$$

for $t_0 \leqslant t < t_1$.

Proof. Let us show that the maximal solution $r(t)$ of (2.2.7) is identically zero. Suppose, on the contrary, that there exists a σ, $t_0 < \sigma < t_0 + a$, such that $r(\sigma) > 0$. Because of the inequality (2.2.6), we have

$$r'(t) \leqslant g(t, r(t)), \qquad t_0 < t \leqslant \sigma.$$

If $\rho(t)$ is the minimal solution of

$$u' = g(t, u), \qquad u(\sigma) = r(\sigma),$$

an application of Theorem 1.4.6 shows that

$$\rho(t) \leqslant r(t), \tag{2.2.8}$$

as far as $\rho(t)$ exists to the left of σ. The solution $\rho(t)$ can be continued to $t = t_0$. If $\rho(\tau) = 0$, for some τ, $t_0 < \tau < \sigma$, we can effect the continuation by defining $\rho(t) = 0$ for $t_0 < t < \tau$. Otherwise, (2.2.8) ensures the possibility of continuation. Since $r(t_0) = 0$, $\lim_{t \to t_0+} \rho(t) = 0$, and we define $\rho(t_0) = 0$. Furthermore, since $g_1(t, u)$ is continuous at $(t_0, 0)$ and $g_1(t_0, 0) = 0$, $r'_+(t_0)$ exists and is equal to zero. This, because of (2.2.8), implies that $\rho'_+(t_0)$ exists and $\rho'_+(t_0) = 0$. But we have assumed that $g(t, u)$ satisfies hypothesis (i) of Theorem 2.2.2. Hence, $\rho(t) \equiv 0$. This contradicts the fact that $\rho(\sigma) = r(\sigma) > 0$. Therefore, $r(t) \equiv 0$, and the proof is complete.

COROLLARY 2.2.2. The function $g(t, u) = \lambda(t)u$, where $\lambda(t) \geqslant 0$ is continuous on $t_0 < t \leqslant t_0 + a$, satisfies the requirements of Theorem 2.2.3, provided that

$$\limsup_{t \to t_0+} [1 + \lambda(t)]e^{-p(t)} > 0, \tag{2.2.9}$$

where

$$p(t) = \int_t^{t_0} \lambda(s)\, ds, \qquad t \neq t_0 . \tag{2.2.10}$$

Proof. Consider the differential equation

$$u' = \lambda(t)u. \tag{2.2.11}$$

The solutions $u(t) \not\equiv 0$ of (2.2.11) are nonvanishing constant multiples of the function $e^{-p(t)}$, $p(t)$ being given by (2.2.10). The derivative of this function is $\lambda(t)e^{-p(t)}$. Since $\lambda(t) \geqslant 0$, it follows from assumption (2.2.9) that every solution $u(t) \not\equiv 0$ of (2.2.11) violates at least one of the two limiting conditions (2.2.5). Hence, the function $g(t, u) = \lambda(t)u$ satisfies hypothesis (i) of Theorem 2.2.2.

COROLLARY 2.2.3. Let the assumptions of Theorem 2.2.3 hold except that the function $g_1(t, u)$ is continuous on $t_0 < t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$. Then, for every t_1, $t_0 < t_1 < t_0 + a$, $u(t) \equiv 0$ is the only function differentiable on $t_0 < t < t_1$ and continuous on $t_0 \leqslant t < t_1$, for which $u'_+(t_0)$ exists,

$$u'(t) = g_1(t, u(t)), \qquad t_0 < t < t_1 ,$$

and (2.2.5) holds.

Proof. We have to show that all the solutions $u(t) \not\equiv 0$ are such that they violate the limiting properties (2.2.5). Assuming the contrary and proceeding as in the proof of Theorem 2.2.3, it is easy to prove the stated result.

Proof of Theorem 2.2.2. Define the function

$$g_f(t, u) = \sup_{\|x-y\|=u} \|f(t, x) - f(t, y)\| \tag{2.2.12}$$

for $t_0 \leqslant t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$. Since $f(t, x)$ is continuous on R_0, $g_f(t, u)$ is continuous on $t_0 \leqslant t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$. It is clear that condition (2.2.2) holds for the function $g_f(t, u)$ because of (2.2.12). Moreover,

$$g_f(t, u) \leqslant g(t, u)$$

for $t_0 < t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$. Theorem 2.2.3 is now applicable with $g_1(t, u) = g_f(t, u)$, and therefore $g_f(t, u)$ satisfies the assumptions of Theorem 2.2.1. This establishes Theorem 2.2.2.

COROLLARY 2.2.4. The conclusion of Theorem 2.2.2 holds if condition (2.2.2) is replaced by

$$\|f(t, x) - f(t, y)\| \leqslant \frac{\| x - y \|}{t - t_0}$$

for (t, x), $(t, y) \in R_0$, $t \neq t_0$.

COROLLARY 2.2.5. If $t_0 = 0$, then $g(t, u) = \lambda(t)\phi(u)$ is admissible in Theorem 2.2.2 provided that $\lambda(t) \geqslant 0$ is continuous for $0 < t \leqslant a$; $\phi(u)$ is continuous for $u \geqslant 0$ and $\phi(0) = 0$, $\phi(u) > 0$ for $u > 0$; and

$$\int_{0^+} \lambda(s)\, ds < \infty, \qquad \int_{0^+} ds/\phi(s) = \infty.$$

The following is yet another criterion of uniqueness of solutions which generalizes the earlier ones. The statement of the results involves the existence of two controlling functions.

THEOREM 2.2.4. Assume that (i) the functions $A(t)$, $B(t)$ are continuous and nonnegative on $t_0 \leqslant t \leqslant t_0 + a$ such that $A(t_0) = B(t_0) = 0$, $B(t) > 0$, $t > t_0$, and

$$\lim_{t \to t_0^+} A(t)/B(t) = 0; \tag{2.2.13}$$

(ii) the functions $g_1(t, u)$, $g_2(t, u)$ are continuous and nonnegative for $t_0 < t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$; (iii) all the solutions $u(t)$ of

$$u' = g_1(t, u) \tag{2.2.14}$$

with $u(t_0) = 0$ obey

$$u(t) \leqslant A(t), \qquad t_0 \leqslant t \leqslant t_0 + a;$$

(iv) the only solution $v(t)$ of

$$v' = g_2(t, v) \tag{2.2.15}$$

on $t_0 \leqslant t \leqslant t_0 + a$ such that

$$\lim_{t \to t_0^+} v(t)/B(t) = 0 \tag{2.2.16}$$

is the trivial solution; (v) $f \in C[R_0, R^n]$, and, for (t, x), $(t, y) \in R_0$, $t \neq t_0$,

$$\|f(t, x) - f(t, y)\| \leqslant \begin{cases} g_1(t, \|x - y\|), \\ g_2(t, \|x - y\|). \end{cases} \tag{2.2.17}$$

Then, the differential system has at most one solution on $t_0 \leqslant t \leqslant t_0 + a$.

Before proceeding to the proof of Theorem 2.2.4, it is convenient to prove the following:

THEOREM 2.2.5. Let the functions $A(t)$, $B(t)$, $g_1(t, u)$, and $g_2(t, u)$ fulfill hypotheses (i), (ii), (iii), and (iv) of Theorem 2.2.4. Suppose that the function $g(t, u)$ is continuous and nonnegative for $t_0 \leqslant t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$, $g(t, 0) \equiv 0$, and

$$g(t, u) \leqslant \begin{cases} g_1(t, u), \\ g_2(t, u), \end{cases} \qquad t \neq t_0. \tag{2.2.18}$$

Then, $u(t) \equiv 0$ is the only differentiable function on $t_0 \leqslant t \leqslant t_0 + a$ which satisfies

$$u' = g(t, u), \qquad u(t_0) = 0 \tag{2.2.19}$$

for $t_0 \leqslant t \leqslant t_0 + a$.

Proof. We shall show that the maximal solution $r(t)$ of (2.2.19) is identically zero. Assuming, on the contrary, that there exists a σ such that $r(\sigma) > 0$ and proceeding as in the proof of Theorem 2.2.3, making use of relations (2.2.18) and (2.2.19), we obtain

$$\rho_2(t) \leqslant r(t), \tag{2.2.20}$$

as far as $\rho_2(t)$ exists to the left of σ, where $\rho_2(t)$ is the minimal solution of (2.2.15) such that $\rho_2(\sigma) = r(\sigma)$. As before, we can continue $\rho_2(t)$ up to t_0 by defining $\rho_2(t_0) = 0$. Since $\rho_2(t) \not\equiv 0$, we have

$$\lim_{t \to t_0^+} \rho_2(t)/B(t) \neq 0,$$

which, in view of (2.2.20), implies that

$$\lim_{t \to t_0^+} r(t)/B(t) \neq 0.$$

This, together with assumption (2.2.13), shows that there exists a t_1 such that

$$r(t_1) > A(t_1). \tag{2.2.21}$$

Let $\rho_1(t)$ be the minimal solution of (2.2.14) such that $\rho_1(t_1) = r(t_1)$. Then it can be shown, arguing similarly, that $\rho_1(t)$ can be continued up to t_0, $\rho_1(t_0) = 0$, and

$$0 \leqslant \rho_1(t) \leqslant r(t), \qquad t_0 \leqslant t \leqslant t_1 . \tag{2.2.22}$$

Since, by hypothesis (iii), all solutions $u(t)$ with $u(t_0) = 0$ of (2.2.14) must obey $u(t) \leqslant A(t)$, $t_0 \leqslant t \leqslant t_0 + a$, we must have

$$\rho_1(t) \leqslant A(t), \qquad t_0 \leqslant t \leqslant t_0 + a.$$

This is absurd because of (2.2.21) and the fact that $\rho_1(t_1) = r(t_1)$. Hence, $\rho_1(t_0) > 0$, which implies, in view of (2.2.22), that

$$0 < \rho_1(t_0) \leqslant r(t_0),$$

contradicting the assumption $r(t_0) = 0$.

Proof of Theorem 2.2.4. Consider the function $g(t, u) = g_f(t, u)$, where $g_f(t, u)$ is the function defined by (2.2.12). By combining the respective arguments in the proofs of Theorems 2.2.2 and 2.2.5, it is easy to show that $g_f(t, u)$ verifies Perron's uniqueness conditions of Theorem 2.2.1, which is sufficient to establish the uniqueness of solutions.

REMARK 2.2.1. Whenever $f(t, x)$ is assumed to be continuous on R_0, it follows from the foregoing considerations that the uniqueness conditions of Theorems 2.2.2 and 2.2.4 can be reduced to that of Perron's condition.

If the pair of functions $g_1(t, u)$, $g_2(t, u)$ satisfies the hypotheses of Theorem 2.2.4, we can also show that there exists a function $g(t, u)$

that fulfills the uniqueness criteria of Kamke as given in Theorem 2.2.2. This is the content of the following:

THEOREM 2.2.6. Let the functions $A(t)$, $B(t)$, $g_1(t, u)$, and $g_2(t, u)$ satisfy hypotheses (i), (ii), (iii), and (iv) of Theorem 2.2.4. Then, there exists a function $g(t, u)$ verifying assumption (i) of Theorem 2.2.2.

Proof. Define the function $g(t, u)$ by

$$g(t, u) = \min[g_1(t, u), g_2(t, u)]. \tag{2.2.23}$$

Then g satisfies (2.2.18). To prove the stated result, it is enough to show that no nontrivial solution of (2.2.4) fulfills the limiting conditions (2.2.5). In fact, the assumption that there exists a differentiable function $u(t)$ satisfying the differential equation (2.2.4) and the conditions (2.2.5) for which $u(\sigma) > 0$, $t_0 < \sigma < t_0 + a$, leads, following the proof of Theorem 2.2.5, to the contradiction that $u(t_0) > 0$.

COROLLARY 2.2.6. The functions $g_1(t, u) = K_1 u^\alpha$, $g_2(t, u) = K_2(u/t)$ are admissible in Theorem 2.2.4, if $0 < \alpha < 1$, $K_2(1 - \alpha) < 1$, with $A(t) = K_1(1 - \alpha)t^{1/(1-\alpha)}$, and $B(t) = t^{K_2}$.

We shall now show that, if certain conditions of Theorem 2.2.2 are violated, Eq. (2.2.3) has nonunique solution. We prove this for the case $n = 1$ and $t_0 = 0$.

THEOREM 2.2.7. Let $g(t, u)$ be continuous on $0 < t \leqslant a$, $0 \leqslant u \leqslant b$, $g(t, 0) \equiv 0$, and $g(t, u) > 0$ for $u > 0$. Suppose that, for each t_1, $0 < t_1 < a$, $u(t) \not\equiv 0$ is a differentiable function on $0 < t < t_1$, and continuous on $0 \leqslant t < t_1$ for which $u'_+(0)$ exists,

$$u' = g(t, u), \qquad 0 < t < t_1,$$

and

$$u(0) = u'_+(0) = 0.$$

Let $f \in C[R_0, R]$, where $R_0 : 0 \leqslant t \leqslant a$, $|x| \leqslant b$, and, for (t, x), $(t, y) \in R_0$, $t \neq 0$,

$$|f(t, x) - f(t, y)| \geqslant g(t, |x - y|). \tag{2.2.24}$$

Then, the scalar differential equation

$$x' = f(t, x), \qquad x(0) = 0 \tag{2.2.25}$$

has at least two solutions on $0 \leqslant t \leqslant a$.

Proof. Let us first suppose that $f(t, 0) \equiv 0$, so that, putting $y = 0$, we obtain the inequality

$$|f(t, x)| \geqslant g(t, |x|),$$

because of condition (2.2.24). Since $f(t, x)$ is continuous and $g(t, u) > 0$ for $u > 0$, it follows that either $f(t, x) < 0$ or $f(t, x) > 0$, for $x \neq 0$. This implies that either

$$f(t, x) \geqslant g(t, |x|) \tag{2.2.26}$$

or

$$f(t, x) \leqslant -g(t, |x|). \tag{2.2.27}$$

By hypothesis, there exists a σ, $0 < \sigma < a$, such that $u(\sigma) > 0$. Let $y(t)$ be the minimal solution of $x' = f(t, x)$, $y(\sigma) = u(\sigma)$. Then, using an argument similar to that in the proof of Theorem 2.2.3 and the inequality (2.2.26), it can be shown that $y(t) \leqslant u(t)$ to the left of σ, as far as $y(t)$ exists. Moreover, $y(t)$ can be continued up to $t = 0$ and

$$0 \leqslant y(t) \leqslant u(t), \qquad 0 < t \leqslant \sigma.$$

Since $u(0) = u'_+(0) = 0$, we obtain from the foregoing relation $y(0) = y'_+(0) = 0$. This proves that the differential equation (2.2.25) has a solution $y(t)$, not identically zero. On the other hand, since $f(t, 0) \equiv 0$, (2.2.25) admits the identically zero solution. Hence, we have two different solutions for Eq. (2.2.25). Corresponding to the case (2.2.27), we can employ a similar reasoning to arrive at the same conclusion.

We shall now remove the restriction $f(t, 0) \equiv 0$. Let $x_0(t)$ be a solution of (2.2.25), existing on $0 \leqslant t \leqslant a$. Using the transformation $z = x - x_0(t)$, we get

$$\begin{aligned} z' &= x' - x_0'(t) = f(t, x) - f(t, x_0(t)) \\ &= f(t, z + x_0(t) - f(t, x_0(t)) \\ &= F(t, z). \end{aligned} \tag{2.2.28}$$

Evidently, $F(t, 0) \equiv 0$. It follows, therefore, that $z \equiv 0$ is one solution of (2.2.28) through $(0, 0)$. But the previous considerations show that (2.2.28) has a solution $z(t)$ different from the identically zero solution. This implies that $x(t) = z(t) + x_0(t)$ is not identically equal to $x_0(t)$, and the theorem is proved.

We have assumed the continuity of $f(t, x)$ in the uniqueness results

that are discussed so far. We shall consider, in what follows, the differential system

$$x' = f(t, x), \qquad x(0) = 0, \tag{2.2.29}$$

where $f(t, x)$ is defined on $R_0 : 0 \leqslant t \leqslant a, \| x \| \leqslant b$. A solution of (2.2.29) in the classical sense will mean a function $x(t)$ that is continuous on $0 \leqslant t \leqslant a$, differentiable on $0 < t < a$, and that satisfies (2.2.29) for $0 < t < a$.

Suppose that $x(t)$, $y(t)$ are two solutions of (2.2.29), existing on $0 \leqslant t \leqslant a$; then, the requirement

$$\lim_{t \to 0^+} \frac{\| x(t) - y(t) \|}{t} = 0,$$

which is satisfied when $f(t, x)$ is continuous at $(0, 0)$, is a necessary condition for the uniqueness of solutions. This can be generalized by the following:

$$\lim_{t \to 0^+} \frac{\| x(t) - y(t) \|}{B(t)} = 0, \tag{2.2.30}$$

where the function $B(t)$ is continuous, positive on $0 < t \leqslant a$, and $B(0^+) = 0$. That condition (2.2.30) is not sufficient for uniqueness is seen by the following:

LEMMA 2.2.1. Suppose that the function $B(t)$ is continuous and positive on $0 < t \leqslant a$ such that $B(0^+) = 0$. Then there exists an infinity of functions $f(t, x)$ such that (2.2.29) has more than one solution satisfying the condition (2.2.30).

Proof. We first construct a function $A(t)$ having a nonnegative derivative $0 \leqslant t \leqslant a$ and

$$\lim_{t \to 0^+} A(t)/B(t) = 0.$$

We proceed as follows. Divide the interval $[0, a]$ into subintervals I_n such that

$$I_1 = (a/2, a), \qquad I_2 = (a/4, a/2), \ldots .$$

Suppose that $b_n = \inf_{I_n} B(t)$. Find a positive linear function L_1 on I_1 such that

$$L_1(a) = b_1, \qquad L_1(a/2) \leqslant \tfrac{1}{2} I_1(a).$$

Then find L_2 on I_2 such that

$$\begin{aligned} L_2(a/2) &= b_2/2 && \text{if} \quad b_2/2 \leqslant L_1(a/2), \\ L_2(a/2) &\leqslant \tfrac{1}{2}L_1(a/2) && \text{if} \quad b_2/2 > L_1(a/2), \\ L_2(a/4) &\leqslant \tfrac{1}{2}L_2(a/2). \end{aligned}$$

We continue this process and then connect the linear functions near the points $a/2^n$ by suitable functions having nonnegative derivatives (for example, by arcs of parabolas). This modification gives us the function $A(t)$ with the required properties.

Having constructed the function $A(t)$, it is easy to define $f(t, x)$ by

$$f(t, x) = x^\alpha A'(t) \qquad (0 < \alpha < 1).$$

Then, $x(t) = 0$ and $x(t) = (1 - \alpha)^{1/(1-\alpha)}[A(t)]^{1/(1-\alpha)}$ are solutions of (2.2.29). It is clear that any two solutions satisfy the condition (2.2.30), and, hence we have the proof.

It is also easy to prove the following fact.

LEMMA 2.2.2. Suppose that $f(t, x)$ is defined on R_0 and is continuous at (0, 0). Then there exists a function $B(t)$ on $0 \leqslant t \leqslant 1$ such that (2.2.30) is satisfied.

Proof. Let $x(t)$, $y(t)$ be two solutions of (2.2.29) on $0 \leqslant t \leqslant a$. Define

$$m(t) = \| x(t) - y(t) \|.$$

Observe that $m(0) = 0$, and, because of the assumed continuity of $f(t, x)$ at (0, 0), we have

$$\lim_{t \to 0^+} m(t)/t = 0.$$

Setting $B(t) = \sup_{s \leqslant t \leqslant 1} m(s)/s$, it is easily verified that

$$\lim_{t \to 0^+} m(t)/B(t) = 0.$$

This is possible if $m(s) \neq 0$ in some neighborhood of the origin; otherwise, the existence of $B(t)$ is trivial.

We notice that the continuity requirement of $f(t, x)$ at (0, 0) is stronger than the condition (2.2.30). To see this, define $f(t, x)$ as follows:

$$f(t, x) = \begin{cases} 1, & x > t, \\ x/t, & 0 < x \leqslant t, \\ 0, & x \leqslant 0. \end{cases}$$

The solutions of (2.2.29) are then given by $x(t) = kt$, where $0 \leqslant k \leqslant 1$. Take $B(t) = t^{1/2}$. Clearly, the relation (2.2.30) is satisfied even though $f(t, x)$ is not continuous at $(0, 0)$.

These considerations lead to

THEOREM 2.2.8. Suppose that $x(t), y(t)$ are any two solutions of (2.2.29) satisfying (2.2.30), where $B(t)$ is positive, continuous on $0 < t \leqslant a$ with $B(0^+) = 0$. Let the function $g(t, u) \geqslant 0$ be continuous on $0 < t \leqslant a$, $0 \leqslant u \leqslant b$, and the only solution $u(t)$ of

$$u' = g(t, u)$$

on $0 \leqslant t \leqslant a$ such that

$$\lim_{t \to 0^+} u(t)/B(t) = 0$$

is the trivial solution. Assume further that the function $f(t, x)$ is defined on R_0 and satisfies

$$\| f(t, x) - f(t, y) \| \leqslant g(t, \| x - y \|)$$

for (t, x), $(t, y) \in R_0$, $t \neq 0$. Then there exists at most one solution of (2.2.29) on $0 \leqslant t \leqslant a$.

Proof. Define $m(t) = \| x(t) - y(t) \|$, where $x(t)$, $y(t)$ are any two solutions of (2.2.29) existing on $0 \leqslant t \leqslant a$. Then $m(0) = 0$, and

$$\begin{aligned} D^+ m(t) &\leqslant \| f(t, x(t)) - f(t, y(t)) \| \\ &\leqslant g(t, m(t)). \end{aligned}$$

If we suppose that, for some σ, $0 < \sigma < a$, $m(\sigma) > 0$, we can show, as in Theorem 2.2.3, that

$$\rho(t) \leqslant m(t), \qquad t \leqslant \sigma,$$

as far as $\rho(t)$ exists, where $\rho(t)$ is the minimal solution of

$$u' = g(t, u), \qquad u(\sigma) = m(\sigma).$$

Furthermore, as before, $\rho(t)$ can be continued up to $t = 0$ and $0 \leqslant \rho(t) \leqslant m(t)$, $0 \leqslant t \leqslant \sigma$. Then, because of the assumed condition (2.2.30), we have

$$0 \leqslant \lim_{t \to 0^+} \rho(t)/B(t) \leqslant \lim_{t \to 0^+} m(t)/B(t) = 0,$$

which, by hypothesis, implies that $\rho(t) \equiv 0$. This contradicts $\rho(\sigma) = m(\sigma) > 0$, and hence $m(t) \equiv 0$, $0 \leqslant t \leqslant a$. The proof is complete.

2.3. Convergence of successive approximations

The answer to the question of whether or not a solution of the system (2.2.3) can always be obtained as a limit of the sequence or a subsequence of the successive approximations is negative. It is not difficult to construct an example such that the solution of (2.2.3) is unique, although no subsequence of the successive approximations converges to that unique solution. It turns out, however, that, with an additional restriction of monotony of $g(t, u)$ in u in Theorems 2.2.1, 2.2.2, and 2.2.4, convergence of successive approximations to the unique solution follows.

Suppose that $g(t, u)$ of Theorem 2.2.2 is monotone nondecreasing in u for fixed t, in addition to the hypotheses of the theorem. Then, defining

$$g_f(t, u) = \sup_{\|x-y\| \leqslant u} \|f(t, x) - f(t, y)\|$$

instead of (2.2.12) for $t_0 \leqslant t \leqslant t_0 + a$, $0 \leqslant u \leqslant 2b$, we note that $g_f(t, u)$ is monotone nondecreasing in u for each t. Thus, it follows that

$$g_f(t, u) \leqslant g(t, u) \qquad (t \neq t_0),$$

and, by Theorem 2.2.3, $g_f(t, u)$ satisfies the hypotheses of Theorem 2.2.1. Similarly, if we assume that $g_1(t, u)$ and $g_2(t, u)$ of Theorem 2.2.4 are monotone nondecreasing in u for each t, then Theorem 2.2.5 shows that $g_f(t, u) \equiv g(t, u)$ also verifies the assumptions of Theorem 2.2.1, in view of the fact that

$$g_f(t, u) \leqslant \begin{cases} g_1(t, u), \\ g_2(t, u), \end{cases} \qquad t \neq t_0 .$$

It is therefore enough to prove the convergence of successive approximations for Theorem 2.2.1 with an additional restriction of monotony on $g(t, u)$.

THEOREM 2.3.1. Let the assumptions of Theorem 2.2.1 hold. Suppose further that $g(t, u)$ is nondecreasing in u for each t, $\|f(t, x)\| \leqslant M$ on R_0, and $\alpha = \min(a, b/M)$. Then, the successive approximations defined by

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds \tag{2.3.1}$$

exist on $t_0 \leqslant t \leqslant t_0 + \alpha$ as continuous functions and converge uniformly on this interval to the solution $x(t)$ of (2.2.3).

Proof. Suppose that $x_k(t)$ is defined and continuous on $t_0 \leqslant t \leqslant t_0 + \alpha$ and satisfies $\| x_k(t) - x_0 \| \leqslant b$ for $k = 0, 1, 2, \ldots, n$. Write

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))\, ds.$$

Then, since $f(t, x_n(t))$ is defined and continuous on $t_0 \leqslant t \leqslant t_0 + \alpha$, the same holds for $x_{n+1}(t)$. Moreover, it is also clear that

$$\begin{aligned} \| x_{n+1}(t) - x_0 \| &\leqslant \int_{t_0}^{t} \| f(s, x_n(s)) \|\, ds \\ &\leqslant M\alpha \leqslant b. \end{aligned}$$

Thus, by induction, the successive approximations are defined and continuous on $t_0 \leqslant t \leqslant t_0 + \alpha$ and

$$\| x_{n+1}(t) - x_0 \| \leqslant b, \qquad n = 0, 1, 2, \ldots.$$

We shall now define the successive approximations for Eq. (2.2.1) as follows:

$$\begin{aligned} u_0(t) &= M(t - t_0), \\ u_{n+1}(t) &= \int_{t_0}^{t} g(s, u_n(s))\, ds, \qquad t_0 \leqslant t \leqslant t_0 + \alpha. \end{aligned}$$

Then,

$$\begin{aligned} \| x_1(t) - x_0 \| &\leqslant \int_{t_0}^{t} \| f(s, x_0) \|\, ds \\ &\leqslant M(t - t_0) = u_0(t). \end{aligned} \tag{2.3.2}$$

Assume that

$$\| x_k(t) - x_{k-1}(t) \| \leqslant u_{k-1}(t) \tag{2.3.3}$$

for a given k. Since

$$\| x_{k+1}(t) - x_k(t) \| \leqslant \int_{t_0}^{t} \| f(s, x_k(s)) - f(s, x_{k-1}(s)) \|\, ds,$$

the inequalities (2.2.2), (2.3.3), and the monotonic character of $g(t, u)$ in u give

$$\| x_{k+1}(t) - x_k(t) \| \leqslant \int_{t_0}^{t} g(s, u_{k-1}(s))\, ds = u_k(t).$$

Thus, by induction, the inequality

$$\| x_{n+1}(t) - x_n(t) \| \leqslant u_n(t) \tag{2.3.4}$$

is true for all n. Furthermore,

$$
\begin{aligned}
\| x'_{n+1}(t) - x'_n(t)\| &\leqslant \|f(t, x_n(t)) - f(t, x_{n-1}(t))\| \\
&\leqslant g(t, \| x_n(t) - x_{n-1}(t)\|) \\
&\leqslant g(t, u_{n-1}(t)).
\end{aligned} \tag{2.3.5}
$$

Let $1 \leqslant n \leqslant m$. Then, using (2.2.2) and (2.3.5), we get

$$
\begin{aligned}
\| x'_n(t) - x'_m(t)\| &\leqslant \| x'_n(t) - x'_{n+1}(t)\| + \| x'_m(t) - x'_{m+1}(t)\| \\
&\quad + \| x'_{n+1}(t) - x'_{m+1}(t)\| \\
&\leqslant g(t, u_{n-1}(t)) + g(t, u_{m-1}(t)) \\
&\quad + g(t, \| x_n(t) - x_m(t)\|).
\end{aligned}
$$

Since by Lemma 1.3.2 we have

$$
u_{n+1}(t) \leqslant u_n(t) \qquad \text{for all} \quad n,
$$

it follows that

$$
D^+\| x_n(t) - x_m(t)\| \leqslant g(t, \| x_n(t) - x_m(t)\|) + 2g(t, u_{n-1}(t))
$$

because of the monotonicity of $g(t, u)$ in u. An application of Theorem 1.4.1 yields that

$$
\| x_n(t) - x_m(t)\| \leqslant r_n(t), \qquad t_0 \leqslant t \leqslant t_0 + \alpha,
$$

where $r_n(t)$ is the maximal solution of

$$
y' = g(t, y) + 2g(t, u_{n-1}(t)), \qquad y_n(t_0) = 0
$$

for each n. Since the conditions of Lemma 1.3.2 are satisfied $r_n(t) \to 0$ uniformly on $t_0 \leqslant t \leqslant t_0 + \alpha$, as $n \to \infty$. This implies that $x_n(t)$ converges uniformly to $x(t)$ on $t_0 \leqslant t \leqslant t_0 + \alpha$ as $n \to \infty$. By Theorem 2.2.1, the solution of (2.2.3) being unique, this $x(t)$ is the unique solution of (2.2.3).

Another proof of Theorem 2.3.1. It can be easily shown that the sequence of approximations (2.3.1) is uniformly bounded and equicontinuous on $t_0 \leqslant t \leqslant t_0 + \alpha$, and therefore there exists uniformly convergent subsequences. Suppose that $x_n(t) - x_{n-1}(t) \to 0$ as $n \to \infty$; then (2.3.1) implies that the limit of any such subsequence is the unique solution $x(t)$ of (2.2.3). It then follows that a selection of a subsequence is unnecessary and that the full sequence $x_0(t)$, $x_1(t)$, $x_2(t)$,... converges uniformly to $x(t)$.

Thus, to prove Theorem 2.3.1, it is sufficient to show that $m(t) \equiv 0$, where

$$m(t) = \limsup_{n\to\infty} \| x_n(t) - x_{n-1}(t)\|. \tag{2.3.6}$$

We shall first show that $m(t)$ is continuous for $t \in [t_0, t_0 + \alpha]$. Since $\|f(t, x)\| \leqslant M$ on R_0, we see that

$$\begin{aligned}\| x_n(t_1) - x_{n-1}(t_1)\| &\leqslant \| x_n(t_2) - x_{n-1}(t_2)\| + 2M| t_1 - t_2 | \\ &\leqslant m(t_2) + 2M| t_1 - t_2 | + \epsilon\end{aligned}$$

for large n, if $\epsilon > 0$. Hence, we have

$$m(t_1) \leqslant m(t_2) + 2M| t_1 - t_2 | + \epsilon.$$

As t_1, t_2 can be interchanged and $\epsilon > 0$ is arbitrary, we obtain

$$| m(t_1) - m(t_2)| \leqslant 2M| t_1 - t_2 |,$$

which proves the continuity of $m(t)$.

The assumption (2.2.2), together with the relation (2.3.1), yields

$$\| x_{n+1}(t) - x_n(t)\| \leqslant \int_{t_0}^{t} g(s, \| x_n(s) - x_{n-1}(s)\|) \, ds.$$

For a fixed t in the interval $(t_0, t_0 + \alpha]$, there is a sequence of integers $n_1 < n_2 < \cdots$ such that $\| x_{n+1}(t) - x_n(t) \|$ tends to $m(t)$ as $n = n_k \to \infty$, and that

$$m^*(s) = \lim_{n=n_k\to\infty} \| x_n(s) - x_{n-1}(s)\|$$

exists uniformly on $t_0 \leqslant s \leqslant t_0 + \alpha$. Thus,

$$m(t) \leqslant \int_{t_0}^{t} g(s, m^*(s)) \, ds. \tag{2.3.7}$$

Since g is assumed to be monotone nondecreasing in u and $m^*(s) \leqslant m(s)$, we obtain from (2.3.7) the inequality

$$m(t) \leqslant \int_{t_0}^{t} g(s, m(s)) \, ds.$$

By Theorem 1.9.2, $m(t) \leqslant r(t)$, where $r(t)$ is the maximal solution of (2.2.1). As Eq. (2.2.1) is assumed to possess only identically zero solution, it follows that $r(t) \equiv 0$, which in turn shows that $m(t) \equiv 0$ on $[t_0, t_0 + \alpha]$. This completes the proof.

2.4. Chaplygin's method

We are interested in establishing a method of approximation of the solution of a given differential equation by means of solutions of an associated linear equation and in estimating the difference between them. This is precisely what the Chaplygin's method accomplishes. For convenience, we shall first consider the case of scalar differential equation.

THEOREM 2.4.1. Let $f \in C[R_0, R]$, where R_0 is the rectangle $t_0 \leqslant t \leqslant t_0 + a$, $|x - x_0| \leqslant b$. Let $|f(t, x)| \leqslant M$ on R_0 and $\alpha = \min(a, b/M)$. Suppose that f_x, f_{xx} exist and $f_{xx} > 0$ in R_0. Let the functions $u_0 = u_0(t)$, $v_0 = v_0(t)$ be differentiable for $t_0 \leqslant t \leqslant t_0 + \alpha$ such that $(t, u_0(t))$, $(t, v_0(t)) \in R_0$ and

$$u_0'(t) < f(t, u_0(t)), \qquad u_0(t_0) = x_0, \tag{2.4.1}$$

$$v_0'(t) > f(t, v_0(t)), \qquad v_0(t_0) = x_0. \tag{2.4.2}$$

Then, there exists a Chaplygin sequence $\{u_n(t), v_n(t)\}$ such that

$$u_n(t) < u_{n+1}(t) < x(t) < v_{n+1}(t) < v_n(t), \qquad t \in [t_0, t_0 + \alpha],$$
$$u_n(t_0) = x_0 = v_n(t_0),$$

where $x(t)$ is the unique solution of

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{2.4.3}$$

existing on $[t_0, t_0 + \alpha]$. Also, $u_n(t)$ and $v_n(t)$ tend uniformly to $x(t)$ on $[t_0, t_0 + \alpha]$ as $n \to \infty$. If, in addition, for a suitable constant β,

$$0 \leqslant v_0(t) - u_0(t) \leqslant \beta,$$

then

$$|u_n(t) - v_n(t)| \leqslant 2\beta/2^{2^n}, \qquad t \in [t_0, t_0 + \alpha]. \tag{2.4.4}$$

Proof. The functions $u_0(t)$, $v_0(t)$, and $x(t)$ satisfy the assumptions of Theorem 1.2.3, and therefore we have

$$u_0(t) < x(t) < v_0(t), \qquad t \in (t_0, t_0 + \alpha].$$

We now define the functions

$$f_1(t, x; u_0, v_0) = f(t, u_0(t)) + f_x(t, u_0(t))(x - u_0(t)), \tag{2.4.5}$$

$$f_2(t, x; u_0, v_0) = f(t, u_0(t)) + \frac{f(t, u_0(t)) - f(t, v_0(t))}{u_0(t) - v_0(t)}(x - u_0(t)). \tag{2.4.6}$$

Observe that, for $t = t_0$,

$$f_1(t_0, x; u_0, v_0) = f_2(t_0, x; u_0, v_0).$$

Let $u_1(t)$, $v_1(t)$ be the solutions of the linear differential equations

$$u_1'(t) = f_1(t, u_1(t); u_0, v_0), \qquad u_1(t_0) = x_0, \tag{2.4.7}$$

$$v_1'(t) = f_2(t, v_1(t); u_0, v_0), \qquad v_1(t_0) = x_0, \tag{2.4.8}$$

which exist on $[t_0, t_0 + \alpha]$. From the inequality (2.4.1) and the definition of f_1 in (2.4.5) result

$$\begin{aligned} u_0'(t) &< f(t, u_0(t)) \\ &= f_1(t, u_0(t); u_0, v_0), \end{aligned}$$

which, because of Theorem 1.2.1 and the following remark, yields

$$u_0(t) < u_1(t), \qquad t \in (t_0, t_0 + \alpha]. \tag{2.4.9}$$

A similar reasoning with (2.4.2) and (2.4.6) shows that

$$v_1(t) < v_0(t), \qquad t \in (t_0, t_0 + \alpha]. \tag{2.4.10}$$

We shall next show that the functions $u_1(t)$, $v_1(t)$ also satisfy the differential inequalities (2.4.1) and (2.4.2), respectively. Since $f_x(t, x)$ is strictly increasing in x, using (2.4.5), (2.4.7), and the mean value theorem, it is easy to deduce that

$$\begin{aligned} u_1'(t) &= f_1(t, u_1(t); u_0, v_0) \\ &< f(t, u_1(t)), \qquad t \in (t_0, t_0 + \alpha]. \end{aligned} \tag{2.4.11}$$

On the other hand,

$$\begin{aligned} u_0'(t) &< f(t, u_0(t)) \\ &= f_2(t, u_0(t); u_0, v_0), \end{aligned}$$

and consequently, we have, applying Theorem 1.2.1 again,

$$u_0(t) < v_1(t), \qquad t \in (t_0, t_0 + \alpha]. \tag{2.4.12}$$

Furthermore, it is readily seen that

$$f_x(t, u_0(t)) < \frac{f(t, u_0(t)) - f(t, v_0(t))}{u_0(t) - v_0(t)} \tag{2.4.13}$$

and

$$f(t, v_1(t)) = f(t, u_0(t)) + f_x(t, u_0(t))[v_1(t) - u_0(t)] \\ + \tfrac{1}{2} f_{xx}(t, \xi)[v_1(t) - u_0(t)]^2, \qquad u_0(t) < \xi < v_1(t). \quad (2.4.14)$$

The relations (2.4.8), (2.4.11), (2.4.12), (2.4.13), and (2.4.14), together with the repeated applications of mean value theorem and the assumption $f_{xx}(t, \xi) > 0$, imply

$$v_1'(t) = f_2(t, v_1(t); u_0, v_0) \\ > f(t, v_1(t)), \qquad t \in (t_0, t_0 + \alpha]. \quad (2.4.15)$$

Since the functions $u_1(t)$, $v_1(t)$, and $x(t)$ verify the assumptions of Theorem 1.2.3, we obtain

$$u_1(t) < x(t) < v_1(t), \qquad t \in (t_0, t_0 + \alpha],$$

which, in view of the inequalities (2.4.9) and (2.4.10), gives

$$u_0(t) < u_1(t) < x(t) < v_1(t) < v_0(t), \qquad t \in (t_0, t_0 + \alpha]. \quad (2.4.16)$$

The foregoing considerations define a transformation T that assigns to a given couple of functions $(u_0(t), v_0(t))$ a new couple $(u_1(t), v_1(t))$ satisfying the same inequalities (2.4.1) and (2.4.2), respectively, such that (2.4.16) holds. This implies that

$$(u_1, v_1) = T[(u_0, v_0)].$$

It therefore follows that we can apply the transformation T to the couple (u_1, v_1) to get (u_2, v_2). A repeated application of the transformation T provides a well-defined Chaplygin's sequence

$$(u_{n+1}, v_{n+1}) = T[(u_n, v_n)]$$

of functions satisfying the following relations:

(i) $u_n'(t) < f(t, u_n(t)), \qquad u_n(t_0) = x_0$;

(ii) $v_n'(t) > f(t, v_n(t)), \qquad v_n(t_0) = x_0$;

(iii) $u_n(t) < u_{n+1}(t) < x(t) < v_{n+1}(t) < v_n(t), \qquad t \in (t_0, t_0 + \alpha]$;

(iv) $u_{n+1}'(t) = f_1(t, u_{n+1}(t); \; u_n(t), v_n(t))$;

(v) $v_{n+1}'(t) = f_2(t, v_{n+1}(t); \; u_n(t), v_n(t))$.

It is clear from (iii) that the sequences $\{u_n\}$, $\{v_n\}$ are monotonic and uniformly bounded on $[t_0, t_0 + \alpha]$. Furthermore, they are equicontin-

uous, in view of the fact that, for each fixed n, u_n, v_n are solutions of linear equations. Hence, an application of Theorem 1.1.1 proves the uniform convergence of $u_n(t)$, $v_n(t)$ to $x(t)$ as $n \to \infty$. Let

$$K = \sup_{\substack{u_0(t) \leqslant x \leqslant v_0(t) \\ t_0 \leqslant t \leqslant t_0+\alpha}} |f_x(t, x)| \tag{2.4.17}$$

and

$$H = \sup_{\substack{u_0(t) \leqslant x \leqslant v_0(t) \\ t_0 \leqslant t \leqslant t_0+\alpha}} |f_{xx}(t, x)|. \tag{2.4.18}$$

Assume that $0 \leqslant v_0(t) - u_0(t) \leqslant (2H\alpha e^{k\alpha})^{-1} = \beta$. Clearly, (2.4.4) holds for $n = 0$. Suppose it is true for a certain fixed n, i.e.,

$$| u_n(t) - v_n(t)| \leqslant 2\beta/2^{2^n}. \tag{2.4.19}$$

From the definition of $u_{n+1}(t)$, $v_{n+1}(t)$, and the mean value theorem, it follows that

$$\begin{aligned} v'_{n+1}(t) - u'_{n+1}(t) &= \frac{f(t, u_n(t)) - f(t, v_n(t))}{u_n(t) - v_n(t)} [v_{n+1}(t) - u_n(t)] \\ &\quad - f_x(t, u_n(t))[u_{n+1}(t) - u_n(t)] \\ &= f_x(t, \xi)[v_{n+1}(t) - u_{n+1}(t)] \\ &\quad + [u_{n+1}(t) - u_n(t)][f_x(t, \xi) - f_x(t, u_n(t))], \end{aligned} \tag{2.4.20}$$

where $u_n(t) < \xi < v_n(t)$. Since

$$f_x(t, \xi) - f_x(t, u_n(t)) = f_{xx}(t, \eta)[\xi - u_n(t)],$$

where $u_n(t) < \eta < \xi$, we obtain, from (2.4.17), (2.4.18), and (2.4.20),

$$\begin{aligned} | v'_{n+1}(t) - u'_{n+1}(t)| &\leqslant K| v_{n+1}(t) - u_{n+1}(t)| \\ &\quad + H| \xi - u_n(t)|| u_{n+1}(t) - u_n(t)|. \end{aligned} \tag{2.4.21}$$

Furthermore, we also have

$$| \xi - u_n(t)| \leqslant | v_n(t) - u_n(t)|$$

and

$$| u_{n+1}(t) - u_n(t)| \leqslant | v_n(t) - u_n(t)|.$$

These estimates, together with (2.4.19) and (2.4.21), lead to the differential inequality

$$D^+| v_{n+1}(t) - u_{n+1}(t)| \leqslant K| v_{n+1}(t) - u_{n+1}(t)| + H \frac{2^2\beta^2}{2^{2^{n+1}}},$$

which, in view of Theorem 1.4.1, yields

$$| v_{n+1}(t) - u_{n+1}(t)| \leqslant \frac{H2^2\beta^2}{2^{2^{n+1}}} \int_{t_0}^{t} e^{K(t-s)}\, ds.$$

Since $\int_{t_0}^{t} e^{K(t-s)}\, ds \leqslant \alpha e^{K\alpha}$, we get

$$| v_{n+1}(t) - u_{n+1}(t)| \leqslant \frac{2\beta}{2^{2^{n+1}}}.$$

Thus, by induction, the relation (2.4.4) is true for all n, and consequently we have, by (iii),

$$| x(t) - u_n(t)| \leqslant 2\beta/2^{2^n}$$

and

$$| x(t) - v_n(t)| \leqslant 2\beta/2^{2^n}.$$

This completes the proof.

Let us now consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0. \tag{2.4.22}$$

In this case, we shall be able to demonstrate only the lower Chaplygin's sequence $\{u_n\}$, under some additional restrictions.

THEOREM 2.4.2. Let $f \in C[R_0, R^n]$, where R_0 is the set,

$$R_0 : t_0 \leqslant t \leqslant t_0 + a, \qquad \| x - x_0 \| \leqslant b.$$

Let $\| f(t, x) \| \leqslant M$ on R_0. We suppose that $f(t, x)$ is quasi-monotone nondecreasing in x, for each $t \in [t_0, t_0 + a]$, and that $\partial f(t, x)/\partial x$ exists and is continuous on R_0. Let $u_0(t)$ be continuously differentiable on $[t_0, t_0 + \alpha]$, where $\alpha = \min(a, b/M)$, $(t, u_0(t)) \in R_0$, and $u_0'(t) < f(t, u_0(t))$, $u_0(t_0) = x_0$.

Furthermore, let

$$f(t, x) + f_x(t, x) \cdot (y - x) < f(t, y) \qquad \text{if} \quad x < y. \tag{2.4.23}$$

Then, there exists a Chaplygin sequence $\{u_n(t)\}$ such that $u_n(t_0) = x_0$,

$$u_n(t) < u_{n+1}(t) < x(t), \qquad t \in [t_0, t_0 + \alpha],$$

where $x(t)$ is the solution of (2.4.22) existing on $[t_0, t_0 + \alpha]$, and

$$\lim_{n \to \infty} u_n(t) = x(t)$$

uniformly on $[t_0, t_0 + \alpha]$.

Proof. We notice, first of all, that $\partial f_i/\partial x_j \geqslant 0$ for $i \neq j$. This follows from the quasi-monotonicity of $f(t, x)$. Moreover, applying Corollary 1.5.1, we obtain

$$u_0(t) < x(t), \qquad t \in [t_0, t_0 + \alpha].$$

Corresponding to the linear equation (2.4.7), we have now to consider the linear system defined by

$$\begin{aligned} y' &= f(t, u_0(t)) + \frac{\partial f(t, u_0(t))}{\partial x}[y - u_0(t)] \\ &\equiv \tilde{f}(t, y; u_0(t)), \qquad y(t_0) = x_0 . \end{aligned} \tag{2.4.24}$$

Observe that $\tilde{f}(t, y;\ u_0(t))$ possesses the quasi-monotone property in y because $\partial f_i/\partial x_j \geqslant 0$, $i \neq j$. Hence it follows by Corollary 1.5.1 that

$$u_0(t) < u_1(t), \qquad t \in [t_0, t_0 + \alpha],$$

where $u_1(t)$ is the solution of (2.4.24). The assumption (2.4.23) implies that

$$\begin{aligned} u_1'(t) &= f(t, u_1(t); u_0(t)) \\ &< f(t, u_1(t)), \end{aligned}$$

and an appeal to Corollary 1.5.1 yields

$$u_1(t) < x(t), \qquad t \in [t_0, t_0 + \alpha].$$

Thus, we have

$$u_0(t) < u_1(t) < x(t), \qquad t \in [t_0, t_0 + \alpha].$$

As in Theorem 2.4.1, we can define the transformation verifying

$$u_1(t) = T[u_0(t)].$$

The rest of the argument is but a repetition of the proof of Theorem 2.4.1 with appropriate changes. This establishes the method of Chaplygin for systems.

2.5. Dependence on initial conditions and parameters

We shall consider the problem of continuity and differentiability of solutions $x(t, t_0, x_0)$ of the differential system

$$x' = f(t, x), \tag{2.5.1}$$

with an initial condition

$$x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{2.5.2}$$

with respect to the initial values (t_0, x_0).

LEMMA 2.5.1. Let $f \in C[J \times R^n, R^n]$, and let

$$G(t, r) = \max_{\|x - x_0\| \leqslant r} \|f(t, x)\|.$$

Assume that $r^*(t, t_0, 0)$ is the maximal solution of

$$u' = G(t, u),$$

through $(t_0, 0)$. Let $x(t, t_0, x_0)$ be any solution of (2.5.1) and (2.5.2). Then,

$$\| x(t, t_0, x_0) - x_0 \| \leqslant r^*(t, t_0, 0), \qquad t \geqslant t_0.$$

Proof. Define $m(t) = \| x(t, t_0, x_0) - x_0 \|$. Then,

$$\begin{aligned} D^+m(t) &\leqslant \| x'(t, t_0, x_0)\| \\ &= \|f(t, x(t, t_0, x_0))\| \\ &\leqslant \max_{\|x - x_0\| \leqslant m(t)} \|f(t, x)\| \\ &= G(t, m(t)). \end{aligned}$$

This implies, by Theorem 1.4.1, that

$$m(t) = \| x(t, t_0, x_0) - x_0 \| \leqslant r^*(t, t_0, 0), \qquad t \geqslant t_0,$$

and this proves the lemma.

THEOREM 2.5.1. Let $f \in C[J \times R^n, R^n]$, and, for $(t, x), (t, y) \in J \times R^n$,

$$\|f(t, x) - f(t, y)\| \leqslant g(t, \| x - y \|), \tag{2.5.3}$$

where $g \in C[J \times R_+, R_+]$. Assume that $u(t) \equiv 0$ is the unique solution of the differential equation

$$u' = g(t, u) \tag{2.5.4}$$

such that $u(t_0) = 0$. Then, if the solutions $u(t, t_0, u_0)$ of (2.5.4) through every point (t_0, u_0) are continuous with respect to initial conditions

(t_0, u_0), the solutions $x(t, t_0, x_0)$ of (2.5.1) and (2.5.2) are unique and continuous with respect to the initial values (t_0, x_0).

Proof. Since the uniqueness of solutions follows from Theorem 2.2.1, we have to prove the continuity part only. To that end, let $x(t, t_0, x_0)$, $y(t, t_0, y_0)$ be the solutions of (2.5.1) through (t_0, x_0), (t_0, y_0), respectively. Defining

$$m(t) = \| x(t, t_0, x_0) - y(t, t_0, y_0)\|,$$

the condition (2.5.3) implies the inequality

$$D^+m(t) \leqslant g(t, m(t)),$$

and, by Theorem 1.4.1, we obtain

$$m(t) \leqslant r(t, t_0, \| x_0 - y_0 \|), \qquad t \geqslant t_0,$$

where $r(t, t_0, \| x_0 - y_0 \|)$ is the maximal solution of (2.5.4) such that $u(t_0) = \| x_0 - y_0 \|$. Since the solutions $u(t, t_0, u_0)$ of (2.5.4) are assumed to be continuous with respect to the initial values, it follows that

$$\lim_{x_0 \to y_0} r(t, t_0, \| x_0 - y_0 \|) = r(t, t_0, 0),$$

and, by hypothesis, $r(t, t_0, 0) \equiv 0$. This, in view of the definition of $m(t)$, yields that

$$\lim_{x_0 \to y_0} x(t, t_0, x_0) = y(t, t_0, y_0),$$

which shows the continuity of $x(t, t_0, x_0)$ with respect to x_0.

We shall next prove the continuity with respect to initial time t_0. If $x(t, t_0, x_0)$, $y(t, t_1, x_0)$, $t_1 > t_0$, are the solutions of (2.5.1) through (t_0, x_0), (t_1, x_0), respectively, then, as before, we obtain the inequality

$$D^+m(t) \leqslant g(t, m(t)),$$

where

$$m(t) = \| x(t, t_0, x_0) - y(t, t_1, x_0)\|.$$

Also,

$$m(t_1) = \| x(t_1, t_0, x_0) - x_0 \|.$$

Hence, by Lemma 2.5.1,

$$m(t_1) \leqslant r^*(t_1, t_0, 0),$$

and, consequently,

$$m(t) \leqslant \tilde{r}(t), \qquad t > t_1,$$

where

$$\tilde{r}(t) = \tilde{r}(t, t_1, r^*(t_1, t_0, 0))$$

is the maximal solution of (2.5.4) through $(t_1, r^*(t_1, t_0, 0))$. Since $r^*(t_0, t_0, 0) = 0$, we have

$$\lim_{t_1 \to t_0} \tilde{r}(t, t_1, r^*(t_1, t_0, 0)) = \tilde{r}(t, t_0, 0),$$

and, by hypothesis, $\tilde{r}(t, t_0, 0)$ is identically zero, thus proving the continuity of $x(t, t_0, x_0)$ with respect to t_0.

COROLLARY 2.5.1. The function $g(t, u) = Lu$, $L > 0$, is admissible in Theorem 2.5.1.

THEOREM 2.5.2. Let $f \in C[E, R^n]$, where E is an open (t, x, μ)-set in R^{n+m+1}, and for $\mu = \mu_0$, let $x_0(t) = x(t, t_0, x_0, \mu_0)$ be a solution of

$$x' = f(t, x, \mu_0), \qquad x(t_0) = x_0, \tag{2.5.5}$$

existing for $t \geqslant t_0$. Assume further that

$$\lim_{\mu \to \mu_0} f(t, x, \mu) = f(t, x, \mu_0), \tag{2.5.6}$$

uniformly in (t, x), and, for (t, x_1, μ), $(t, x_2, \mu) \in E$,

$$\| f(t, x_1, \mu) - f(t, x_2, \mu)\| \leqslant g(t, \| x_1 - x_2 \|), \tag{2.5.7}$$

where $g \in C[J \times R_+, R_+]$. Suppose that $u(t) \equiv 0$ is the unique solution of (2.5.4) such that $u(t_0) = 0$. Then, given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for every μ, $\| \mu - \mu_0 \| < \delta(\epsilon)$, the differential system

$$x' = f(t, x, \mu), \qquad x(t_0) = x_0 \tag{2.5.8}$$

admits a unique solution $x(t) = x(t, t_0, x_0, \mu)$ satisfying

$$\| x(t) - x_0(t)\| < \epsilon, \qquad t \geqslant t_0.$$

Proof. The uniqueness of solutions is obvious from Theorem 2.2.1. From the assumption that $u(t) \equiv 0$ is the only solution of (2.5.4), it follows, by Lemma 1.3.1, that, given any compact interval $[t_0, t_0 + a]$ contained in J and any $\epsilon > 0$, there exists a positive number $\eta = \eta(\epsilon)$ such that the maximal solution $r(t, t_0, 0, \eta)$ of

$$u' = g(t, u) + \eta$$

exists on $t_0 \leqslant t \leqslant t_0 + a$ and satisfies

$$r(t, t_0, 0, \eta) < \epsilon, \qquad t \in [t_0, t_0 + a].$$

Furthermore, because of the condition (2.5.6), given $\eta > 0$, there exists a $\delta = \delta(\eta) > 0$ such that

$$\| f(t, x, \mu) - f(t, x, \mu_0) \| < \eta$$

provided

$$\| \mu - \mu_0 \| < \delta.$$

Now, let $\epsilon > 0$ be given, and define

$$m(t) = \| x(t) - x_0(t) \|,$$

where $x(t)$, $x_0(t)$ are the solutions of (2.5.8) and (2.5.5), respectively. Then, using the assumption (2.5.7), we get

$$D^+m(t) \leqslant g(t, m(t)) + \| f(t, x_0(t), \mu) - f(t, x_0(t), \mu_0) \|.$$

From this, it turns out that, whenever $\| \mu - \mu_0 \| < \delta$,

$$D^+m(t) \leqslant g(t, m(t)) + \eta.$$

By Theorem 1.4.1, we have

$$m(t) \leqslant r(t, t_0, 0, \eta), \qquad t \geqslant t_0,$$

and hence

$$\| x(t) - x_0(t) \| < \epsilon, \qquad t \geqslant t_0,$$

provided that

$$\| \mu - \mu_0 \| < \delta.$$

Clearly, δ depends on ϵ since η does. The proof is complete.

LEMMA 2.5.2. Let $f \in C[J \times D, R^n]$, where D is an open, convex set in R^n, and let $\partial f / \partial x$ exist and be continuous. Then,

$$f(t, x_2) - f(t, x_1) = \int_0^1 \left[\frac{\partial f(t, sx_2 + (1 - s)x_1}{\partial x} ds \right] \cdot (x_2 - x_1).$$

Proof. Setting

$$F(s) = f(t, sx_2 + (1 - s)x_1), \qquad 0 \leqslant s \leqslant 1,$$

the convexity of D implies that $F(s)$ is defined. Hence,

$$dF/ds = \left[\frac{\partial f(t, sx_2 + (1-s)x_1)}{\partial x}\right] \cdot (x_2 - x_1). \tag{2.5.9}$$

Since $F(1) = f(t, x_2)$ and $F(0) = f(t, x_1)$, the result follows by integrating (2.5.9) from 0 to 1.

THEOREM 2.5.3. Assume that $f \in C[J \times R^n, R^n]$ and possesses continuous partial derivatives $\partial f/\partial x$ on $J \times R^n$. Let the solution $x_0(t) = x(t, t_0, x_0)$ of (2.5.1) exist for $t \geqslant t_0$, and let

$$H(t, t_0, x_0) = \frac{\partial f(t, x(t, t_0, x_0))}{\partial x}.$$

Then

(i) $\Phi(t, t_0, x_0) = \dfrac{\partial x(t, t_0, x_0)}{\partial x_0}$

exists and is the solution of

$$y' = H(t, t_0, x_0)y \tag{2.5.10}$$

such that $\Phi(t_0, t_0, x_0)$ is the unit matrix;

(ii) $\dfrac{\partial x(t, t_0, x_0)}{\partial t_0}$

exists, is the solution of (2.5.10), and satisfies the relation

$$\frac{\partial x(t, t_0, x_0)}{\partial t_0} = -\Phi(t, t_0, x_0) \cdot f(t_0, x_0), \qquad t \geqslant t_0. \tag{2.5.11}$$

Proof. First we shall prove conclusion (i). Let h be a scalar and $e_k = (e_k{}^1, \ldots, e_k{}^n)$ be the vector such that $e_k{}^j = 0$ if $j \neq k$ and $e_k{}^k = 1$. Then, for small h, let

$$x(t, h) = x(t, t_0, x_0 + e_k h),$$

which is defined on J, and

$$\lim_{h \to 0} x(t, h) = x_0(t)$$

uniformly on J. Since

$$[x(t, h) - x_0(t)]' = f(t, x(t, h)) - f(t, x_0(t)),$$

applying Lemma 2.5.2 with $x_2 = x(t, h)$, $x_1 = x_0(t)$, we have

$$[x(t, h) - x_0(t)]' = \int_0^1 \left[\frac{\partial f(t, sx_2 + (1 - s)x_1)}{\partial x} ds\right] \cdot (x_2 - x_1).$$

If we write

$$x_h(t) = \frac{x(t, h) - x_0(t)}{h}, \qquad h \neq 0,$$

the existence of $\partial x(t, t_0, x_0)/\partial x_0$ is equivalent to the existence of the limit of $x_h(t)$ as $h \to 0$, since $x(t_0, h) = x_0 + e_k h$, $x_h(t_0) = e_k$. Thus, $x_h(t)$ is the solution of the initial value problem

$$y' = H(t, t_0, x_0, h)y, \qquad y(t_0) = e_k, \tag{2.5.12}$$

where

$$H(t, t_0, x_0, h) = \int_0^1 \frac{\partial f(t, x(t, h)s + (1 - s)x_0(t))}{\partial x} ds.$$

As $x(t, h) \to x_0(t)$ as $h \to 0$, by the continuity of $\partial f/\partial x$, it follows that

$$\lim_{h \to 0} H(t, t_0, x_0, h) = H(t, t_0, x_0)$$

uniformly on J.

Considering (2.5.12) as a family of initial value problems depending on a parameter h, where $H(t, t_0, x_0, h)$ is continuous for $t \in J$, h being small and y arbitrary, and observing that the solutions of (2.5.12) are unique, it is clear that the general solution of (2.5.12) is a continuous function of h. In particular, $\lim_{h \to 0} x_h(t) = x(t)$ exists and is the solution of (2.5.10) on J. This implies that $\partial x(t, t_0, x_0)/\partial x_0$ exists and is the solution of (2.5.10).

To prove (ii), define

$$\hat{x}_h(t) = \frac{x(t, t_0 + h, x_0) - x(t, t_0, x_0)}{h}, \qquad h \neq 0.$$

Since (2.5.1) has unique solutions, we have

$$x(t, t_0 + h, x_0) = x(t, t_0, x(t_0, t_0 + h, x_0)),$$

and therefore

$$h\hat{x}_h(t) = x(t, t_0, x(t_0, t_0 + h, x_0)) - x(t, t_0, x_0). \tag{2.5.13}$$

Because $\partial x(t, t_0, x_0)/\partial x_0$ exists and is continuous and

$$x(t_0, t_0 + h, x_0) \to x(t_0, t_0, x_0) = x_0 \qquad \text{as} \quad h \to 0,$$

it follows from (2.5.13) that

$$h\hat{x}_h(t) = \left[\frac{\partial x(t, t_0, x_0)}{\partial x_0} + 0(1)\right] [x(t_0, t_0 + h, x_0) - x_0], \qquad (2.5.14)$$

as $h \to 0$. By the mean value theorem, there exists a $\theta = \theta_k$, $k = 1, 2, \dots, n$ such that

$$x_k(t_0, t_0 + h, x_0) - x_{0,k} = -hf_k(t_0 + \theta h, x(t_0 + \theta h, t_0 + h, x_0)),$$

where $0 < \theta < 1$. Notice that, for each k,

$$f_k(t_0 + \theta h, x(t_0 + \theta h, t_0 + h, x_0)) = f_k(t_0, x_0) + 0(1)$$

as $h \to 0$. Thus, (2.5.14) shows that

$$\hat{x}_h(t) = -\left[\frac{\partial x(t, t_0, x_0)}{\partial x_0} + 0(1)\right] \cdot [f(t_0, x_0) + 0(1)],$$

as $h \to 0$, which implies that $\partial x(t, t_0, x_0)/\partial t_0 = \lim_{h \to 0} \hat{x}_h(t)$ exists and satisfies (2.5.11).

This completes the proof.

2.6. Variation of constants

Let us prove some elementary facts about linear differential systems,

$$x' = A(t)x, \qquad (2.6.1)$$

where $A(t)$ is a continuous $n \times n$ matrix on J. Let $U(t)$ be the $n \times n$ matrix whose columns are the n-vector solutions $x(t)$, $x(t)$, being so chosen to satisfy the initial condition $U(t_0) =$ unit matrix. Since each column of $U(t)$ is a solution of (2.6.1), it is clear that U satisfies the matrix differential equation

$$U' = A(t)U, \qquad U(t_0) = \text{unit matrix}. \qquad (2.6.2)$$

THEOREM 2.6.1. Let $A(t)$ be a continuous $n \times n$ matrix on J. Then the fundamental solution $U(t)$ of (2.6.2) is nonsingular on J. More precisely,

$$\det U(t) = \exp \int_{t_0}^{t} \operatorname{tr} A(s)\, ds, \qquad t \in J,$$

where $\operatorname{tr} A(t) = \sum_{i=1}^{n} a_{ii}(t)$.

Proof. The proof depends on the following two facts:

(i) $d(\det U(t))/dt$ = sum of the determinants formed by replacing the elements of one row of $\det U(t)$ by their derivatives.

(ii) The columns of $U(t)$ are the solutions of (2.6.1).

Simplifying the determinants obtained in (i) by the use of (ii), we get

$$\frac{d}{dt} \det U(t) = \operatorname{tr} A(t) \det U(t).$$

The result follows, since $U(t_0)$ = unit matrix.

THEOREM 2.6.2. Let $y(t)$ be a solution of

$$y' = A(t)y + F(t, y), \tag{2.6.3}$$

where $F \in C[J \times R^n, R^n]$, such that $y(t_0) = y_0$. If $U(t)$ is the matrix solution of (2.6.2), then $y(t)$ satisfies the integral equation

$$y(t) = U(t)y_0 + \int_{t_0}^{t} U(t)U^{-1}(s)F(s, y(s))\, ds, \qquad t \geqslant t_0 . \tag{2.6.4}$$

Proof. Defining $y(t) = U(t)z(t)$ and substituting in (2.6.3), we obtain

$$U'(t)z(t) + U(t)z'(t) = A(t)U(t)z(t) + F(t, y(t)).$$

This, because of (2.6.2), yields

$$z'(t) = U^{-1}(t)F(t, y(t)),$$

whence

$$z(t) = y_0 + \int_{t_0}^{t} U^{-1}(s)F(s, y(s))\, ds.$$

Multiplying this equation by $U(t)$ gives (2.6.4).

COROLLARY 2.6.1. Let $A(t)$ be a continuous $n \times n$ matrix on J such that every solution $x(t)$ of (2.6.1) is bounded for $t \geqslant t_0$. Let $U(t)$ be the fundamental matrix of (2.6.1). Then, $U^{-1}(t)$ is bounded if and only if

$$\operatorname{Re}\left[\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right]$$

is bounded from below.

We shall now consider the nonlinear differential system (2.5.1). The following theorem gives an analog of variation of parameters formula for the solutions $y(t, t_0, x_0)$ of

$$y' = f(t, y) + F(t, y). \tag{2.6.5}$$

THEOREM 2.6.3. Let $f, F \in C[J \times R^n, R^n]$, and let $\partial f/\partial x$ exist and be continuous on $J \times R^n$. If $x(t, t_0, x_0)$ is the solution of (2.5.1) and (2.5.2) existing for $t \geqslant t_0$, any solution $y(t, t_0, x_0)$ of (2.6.5), with $y(t_0) = x_0$, satisfies the integral equation

$$\begin{aligned} y(t, t_0, x_0) &= x(t, t_0, x_0) \\ &\quad + \int_{t_0}^{t} \Phi(t, s, y(s, t_0, x_0))F(s, y(s, t_0, x_0))\, ds \end{aligned} \tag{2.6.6}$$

for $t \geqslant t_0$, where $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$.

Proof. Write $y(t) = y(t, t_0, x_0)$. Then,

$$\begin{aligned} \frac{dx(t, s, y(s))}{ds} &= \frac{\partial x(t, s, y(s))}{\partial s} + \frac{\partial x(t, s, y(s))}{\partial y} \cdot y'(s) \\ &= \Phi(t, s, y(s))[y'(s) - f(s, y(s))], \end{aligned} \tag{2.6.7}$$

using Theorem 2.5.3. Noting that $x(t, t, y(t, t_0, x_0)) = y(t, t_0, x_0)$ and $y'(s) - f(s, y(s)) = F(s, y(s))$, by integrating (2.6.7) from t_0 to t, the desired result (2.6.6) follows.

THEOREM 2.6.4. Let $f \in C[J \times R^n, R^n]$, and $\partial f/\partial x$ exist and be continuous on $J \times R^n$. Assume that $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ are the solutions of (2.5.1) through (t_0, x_0) and (t_0, y_0), respectively, existing for $t \geqslant t_0$, such that x_0, y_0 belong to a convex subset of R^n. Then, for $t \geqslant t_0$,

$$x(t, t_0, y_0) - x(t, t_0, x_0) = \left[\int_0^1 \Phi(t, t_0, x_0 + s(y_0 - x_0))\, ds\right] \cdot (y_0 - x_0). \tag{2.6.8}$$

Proof. Since x_0, y_0 belong to a convex subset of R^n, $x(t, t_0, x_0 + s(y_0 - x_0))$ is defined for $0 \leqslant s \leqslant 1$. Thus,

$$\frac{dx(t, t_0, x_0 + s(y_0 - x_0))}{ds} = \Phi(t, t_0, x_0 + s(y_0 - x_0)) \cdot (y_0 - x_0),$$

and hence the integration from 0 to 1 yields (2.6.8).

2.7. Upper and lower bounds

Consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{2.7.1}$$

and the differential inequality

$$\| y' - f_1(t, y)\| \leqslant \delta(t), \tag{2.7.2}$$

where $f, f_1 \in C[J \times R^n, R^n]$, and $\delta \in C[J, R_+]$.

DEFINITION 2.7.1. By a δ-*approximate solution* of

$$y' = f_1(t, y), \qquad y(t_0) = y_0, \qquad t_0 \geqslant 0 \tag{2.7.3}$$

on $[t_0, \infty)$, we mean a function $y(t)$ such that $y \in C[J, R^n]$, $y'(t)$ exists on $J - S$, S being an at-most countable subset of J, and satisfies (2.7.2) on $J - S$.

THEOREM 2.7.1. Let $g \in C[J \times R_+, R_+]$ and $u, v, \delta \in C[J, R_+]$ such that

$$\begin{aligned} D_-u(t) &< -[g(t, u(t)) + \delta(t)], \\ D_-v(t) &> g(t, v(t)) + \delta(t) \end{aligned} \tag{2.7.4}$$

for $t > t_0$. Let

$$\Omega = [x, y \in R^n : \| x - y \| = u(t) \quad \text{and} \quad \| x - y \| = v(t), t > t_0].$$

Assume that $f, f_1 \in C[J \times R^n, R^n]$, and

$$\| f(t, x) - f_1(t, y)\| \leqslant g(t, \| x - y \|) \tag{2.7.5}$$

for $t > t_0$ and $x, y \in \Omega$. If $x(t)$, $y(t)$ be a solution and a δ-approximate solution of (2.7.1) and (2.7.3), respectively, on $[t_0, \infty)$ such that

$$u(t_0) < \| x_0 - y_0 \| < v(t_0),$$

then

$$u(t) < \| x(t) - y(t)\| < v(t), \qquad t \geqslant t_0. \tag{2.7.6}$$

Proof. If (2.7.6) is not true, the set

$$z = [t \geqslant t_0 : v(t) \leqslant \| x(t) - y(t)\| \leqslant u(t)]]$$

is nonempty. Arguing as in the proof of Theorem 1.2.1, we arrive at a $t_1 > t_0$ such that either

$$\| x(t_1) - y(t_1)\| = u(t_1)$$

or

$$\| x(t_1) - y(t_1)\| = v(t_1).$$

In either case, it follows from the definition of Ω that, at $t = t_1$, $x(t_1)$, $y(t_1) \in \Omega$, and therefore, defining $m(t) = \| x(t) - y(t) \|$, we get

$$\begin{aligned} | m'_+(t_1)| &\leqslant \| x'(t_1) - y'(t_1)\| \\ &\leqslant \| f(t_1, x(t_1)) - f_1(t_1, y(t_1))\| \\ &\quad + \| y'(t_1) - f_1(t_1, y(t_1))\|. \end{aligned}$$

This, together with (2.7.2) and (2.7.5), implies

$$\begin{aligned} -[g(t_1, m(t_1)) + \delta(t_1)] &\leqslant m'_+(t_1) \\ &\leqslant g(t_1, m(t_1)) + \delta(t_1). \end{aligned}$$

A repetition of the rest of the proof of Theorem 1.2.1, with appropriate changes, proves (2.7.6).

THEOREM 2.7.2. Let the assumptions of Theorem 2.7.1 hold except that (2.7.4) and (2.7.5) are replaced, respectively, by

$$\begin{aligned} D\_u(t) &\leqslant -[g(t, u(t)) + \delta(t)], \\ D\_v(t) &\geqslant g(t, v(t) + \delta(t), \end{aligned} \tag{2.7.7}$$

and

$$\| f(t, x) - f_1(t, y)\| \leqslant g(t, \| x - y \|), \tag{2.7.8}$$

for $t > t_0$, $x, y \in R^n$. Suppose further that, for each $\tau \in [t_0, \infty)$ and $t \in [t_0, \tau]$, g satisfies the condition

$$| g(t, u_1) - g(t, u_2)| \leqslant G(\tau + t_0 - t, u_1 - u_2), \qquad u_1 \geqslant u_2, \tag{2.7.9}$$

where $G \in C[J \times R_+, R_+]$ and $r(t) \equiv 0$ is the maximal solution of

$$u' = G(t, u), \qquad u(t_0) = 0. \tag{2.7.10}$$

Then, the inequality (2.7.6) remains valid.

Proof. By a repeated application of Theorem 1.4.3, we can prove (2.7.6). For this purpose, it is enough to see that $g + \delta$ and $-(g + \delta)$ satisfy the condition (1.4.9), in view of (2.7.9). Also, (2.7.8) implies that

$$-[g(t, m(t)) + \delta(t)] \leqslant m'_+(t) \leqslant g(t, m(t)) + \delta(t),$$

for $t > t_0$.

THEOREM 2.7.3. Let $g \in C[J \times R_+, R_+]$, $\delta \in C[J, R_+]$, and $r(t)$, $\rho(t)$ be the maximal and the minimal solutions of

$$\begin{aligned} u' &= g(t, u) + \delta(t), & u(t_0) &= u_0, \\ v' &= -[g(t, v) + \delta(t)], & v(t_0) &= v_0, \end{aligned} \tag{2.7.11}$$

respectively, existing on $[t_0, \infty)$. Let

$$\begin{aligned} \Omega = [x, y \in R^n : \rho(t) - \epsilon_2 &\leqslant \| x - y \| < \rho(t) \\ \text{and} \quad r(t) &\leqslant \| x - y \| < r(t) + \epsilon_1, \ t \geqslant t_0], \end{aligned}$$

where $\epsilon_1, \epsilon_2 > 0$. Assume that, for $t > t_0$, $x, y \in \Omega$,

$$\| f(t, x) - f_1(t, y) \| \leqslant g(t, \| x - y \|). \tag{2.7.12}$$

If $x(t)$, $y(t)$ be a solution and a δ-approximate solution of (2.7.1) and (2.7.3), respectively, on $[t_0, \infty)$ such that

$$v_0 \leqslant \| x_0 - y_0 \| \leqslant u_0,$$

then

$$\rho(t) \leqslant \| x(t) - y(t) \| \leqslant r(t), \qquad t \geqslant t_0. \tag{2.7.13}$$

Proof. Let $\tau \in [t_0, \infty)$. By Lemma 1.3.1, the maximal and the minimal solutions $r(t, \epsilon)$ and $\rho(t, \epsilon)$ of

$$\begin{aligned} u' &= g(t, u) + \delta(t) + \epsilon, & u(t_0) &= u_0 + \epsilon, \\ v' &= -[g(t, v) + \delta(t) + \epsilon], & v(t_0) &= v_0 - \epsilon \end{aligned}$$

exist for sufficiently small $\epsilon > 0$, and

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon),$$

$$\rho(t) = \lim_{\epsilon \to 0} \rho(t, \epsilon)$$

uniformly on $[t_0, \tau]$. In view of this, there exist $\epsilon_1, \epsilon_2 > 0$ such that

$$r(t, \epsilon) < r(t) + \epsilon_1,$$

$$\rho(t, \epsilon) > \rho(t) - \epsilon_2$$

for $t \in [t_0, \tau]$. Furthermore, an application of Theorem 1.2.1 yields that, for $t \in [t_0, \tau]$,

$$r(t) < r(t, \epsilon), \qquad \rho(t) > \rho(t, \epsilon).$$

It now follows that, for $t \in [t_0, \tau]$,

$$\begin{aligned} \rho(t) - \epsilon_2 < \rho(t, \epsilon) < \rho(t), \\ r(t) < r(t, \epsilon) < r(t) + \epsilon_1 . \end{aligned} \tag{2.7.14}$$

To prove (2.7.13), it is enough to show that

$$\rho(t, \epsilon) < \| x(t) - y(t)\| < r(t, \epsilon), \qquad t \in [t_0, \tau]. \tag{2.7.15}$$

Assuming the contrary and arguing as in Theorem 2.7.1, we get either

$$\| x(t_1) - y(t_1)\| = r(t_1, \epsilon)$$

or

$$\| x(t_1) - y(t_1)\| = \rho(t_1, \epsilon).$$

These relations show, because of the inequalities (2.7.14), that, in either case, $x(t_1)$, $y(t_1) \in \Omega$. By following the rest of the standard argument, it is easy to prove (2.7.15). This completes the proof.

REMARK 2.7.1. Evidently, Theorem 2.7.3 holds when the condition (2.7.12) is satisfied for all $x, y \in R^n$ instead of Ω. Similar comment is valid for Theorem 2.7.1 also. The bounds obtained in the foregoing theorems are on a general setup. They include a number of special cases. For instance, if $\delta(t) \equiv 0$, we get the estimates of the difference of solutions of (2.7.1) and (2.7.3), respectively; whereas if, in addition, $f(t, x) \equiv f_1(t, x)$, the same results yield the growth conditions between any two solutions of the system (2.7.1). On the other hand, if $f_1(t, x) \equiv f(t, x)$, error estimates between a solution and a δ-approximate solution of the system (2.7.3) are obtained. Furthermore, if $\delta(t) \equiv 0$ and $f_1(t, x) \equiv 0$, these results provide the upper and lower bounds of solutions of the system (2.7.1).

For future use, the following well-known result is stated as

COROLLARY 2.7.1. Let $f \in C[J \times R^n, R^n]$, and, for $t \geqslant 0$, $x, y \in R^n$,

$$\| f(t, x) - f(t, y)\| \leqslant L(t)\| x - y \| ,$$

where $L \in C[J, R_+]$. Then, for $t \geqslant t_0$,

$$\| x_0 - y_0 \| \exp \left[- \int_{t_0}^{t} L(s)\, ds \right] \leqslant \| x(t) - y(t) \|$$
$$\leqslant \| x_0 - y_0 \| \exp \left[\int_{t_0}^{t} L(s)\, ds \right], \qquad (2.7.16)$$

where $x(t)$, $y(t)$ are any two solutions of the system (2.7.1), through (t_0, x_0), (t_0, y_0), respectively.

In the foregoing results, the upper bounds obtained are increasing functions of t, since the assumptions demand that $g(t, u) \geqslant 0$ and $\delta(t) > 0$, and therefore give very little information about the growth of solutions for large time. We give below a different set of assumptions that yield sharper bounds because the function $g(t, u)$ need not be restricted to be positive.

THEOREM 2.7.4. Let $g \in C[J \times R_+, R]$, $\delta \in C[J, R_+]$, and $r(t)$ be the maximal solution of

$$u' = g(t, u) + \delta(t), \qquad u(t_0) = u_0,$$

existing on $[t_0, \infty)$. Assume that, for $t \in J$, $x, y \in R^n$,

$$\| x - y + h[f(t, x) - f_1(t, y)] \| \leqslant \| x - y \| + hg(t, \| x - y \|) + 0(h), \quad (2.7.17)$$

for all sufficiently small $h > 0$. Then,

$$\| x_0 - y_0 \| \leqslant u_0$$

implies

$$\| x(t) - y(t) \| \leqslant r(t), \qquad t \geqslant t_0, \qquad (2.7.18)$$

$x(t)$, $y(t)$ being a solution and a δ-approximate solution of (2.7.1) and (2.7.3), respectively, existing on $[t_0, \infty)$.

Proof. Consider the function

$$m(t) = \| x(t) - y(t) \|.$$

We have, for small $h > 0$,

$$m(t + h) = \| x(t) + hf(t, x(t)) + \epsilon(h) - y(t + h) \|$$
$$\leqslant \| x(t) - y(t) + h\, [f(t, x(t)) - f_1(t, y(t))] \|$$
$$+ \| y(t + h) - y(t) - hf_1(t, y(t)) \| + \| \epsilon(h) \|,$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. It therefore follows, using (2.7.2) and (2.7.17), that

$$D^+m(t) \leqslant g(t, m(t)) + \delta(t),$$

which, by Theorem 1.4.1, yields the estimate (2.7.18).

2.8. Componentwise bounds

Instead of the differential inequality (2.7.2), we shall be considering a system of differential inequalities given by

$$|y' - f_1(t, y)| \leqslant \delta(t), \tag{2.8.1}$$

where $\delta \in C[J, R_+^n]$. Here and in what follows, we mean by $|x|$ a vector whose components are $|x_1|, |x_2|,..., |x_n|$ for any $x \in R^n$. Note that $\delta(t)$ is a scalar function in (2.7.2), whereas it is a vector in (2.8.1). In this case, the δ-approximate solution of (2.7.3) must satisfy (2.8.1) in place of (2.7.2).

THEOREM 2.8.1. Let $g \in C[J \times R_+^n, R_+^n]$ and possess the quasi-monotone nondecreasing property. Let $u, v, \delta \in C[J, R_+^n]$ such that, for $t > t_0$,

$$\begin{aligned} D_-u(t) &< -[g(t, u(t)) + \delta(t)], \\ D_-v(t) &> g(t, v(t)) + \delta(t). \end{aligned} \tag{2.8.2}$$

Suppose that $f, f_1 \in C[J \times R^n, R^n]$, and, for $t > t_0$, $x, y \in \Omega_i$,

$$|f_i(t, x) - f_{1,i}(t, y)| \leqslant g_i(t, |x - y|), \tag{2.8.3}$$

where

$$\begin{aligned} \Omega_i = [x, y \in R^n : |x_i - y_i| &= u_i(t), \\ |x_i - y_i| &= v_i(t), \quad t > t_0], \qquad i = 1, 2,..., n. \end{aligned}$$

If $x(t)$, $y(t)$ be a solution and a δ-approximate solution of (2.7.1) and (2.7.3), respectively, on $[t_0, \infty)$ such that

$$u(t_0) < |x_0 - y_0| < v(t_0),$$

then

$$u(t) < |x(t) - y(t)| < v(t), \qquad t \geqslant t_0. \tag{2.8.4}$$

Proof. The proof runs parallel to that of Theorem 2.7.1. However, in this situation, the assumption that the set

$$Z = \bigcup_{i=1}^{n} [t \geqslant t_0 : v_i(t) \leqslant | x_i(t) - y_i(t)| \leqslant u_i(t)]$$

is nonempty leads to the existence of an index j, $1 \leqslant j \leqslant n$, and a $t_1 > t_0$ such that either

$$| x_j(t_1) - y_j(t_1)| = u_j(t_1)$$

or

$$| x_j(t_1) - y_j(t_1)| = v_j(t_1),$$

which shows that $x(t_1), y(t_1) \in \Omega_j$. Consequently, as in Theorem 2.7.1, it is easy to show, using (2.8.3), that

$$-[g_j(t_1, m(t_1)) + \delta_j(t_1)] \leqslant m'_{+,j}(t_1) \leqslant g_j(t_1, m(t_1)) + \delta_j(t_1).$$

Making use of the quasi-monotone property of $g(t, u)$ and the arguments of Theorem 1.5.1, we can prove (2.8.4).

The next theorem is analogous to Theorem 2.7.2 for componentwise bounds, the proof of which can be deduced from Theorem 1.7.3, with an observation similar to that of Theorem 2.7.2.

THEOREM 2.8.2. Assume that, in place of (2.8.2) and (2.8.3), we have

$$D_- u(t) \leqslant -[g(t, u(t)) + \delta(t)],$$
$$D_- v(t) \geqslant g(t, v(t)) + \delta(t),$$

and

$$|f_i(t, x) - f_{1,i}(t, y)| \leqslant g_i(t, | x - y |),$$

for $t > t_0$, $x, y \in R^n$, other assumptions being the same as in Theorem 2.8.1. Moreover, let, for each $\tau \in [t_0, \infty)$, $t \in [t_0, \tau]$ and for each $i = 1, 2, \ldots, n$,

$$| g_i(t, u) - g_i(t, \bar{u})| \leqslant G(\tau + t_0 - t, u_i - \bar{u}_i),$$
$$u_i \geqslant \bar{u}_i, \qquad u_j = \bar{u}_j, \qquad i \neq j,$$

where $G \in C[J \times R_+, R_+]$, and $r(t) \equiv 0$ is the maximal solution of (2.7.10). Then, the assertion of Theorem 2.8.1 remains true.

THEOREM 2.8.3. Let $g \in C[J \times R^n_+, R_+^n]$ and possess the quasi-monotone nondecreasing property. Assume that $r(t)$, $\rho(t)$ are the maximal and the minimal solutions of

$$u' = g(t, u) + \delta(t), \qquad u(t_0) = u_0,$$
$$v' = -[g(t, v) + \delta(t)], \qquad v(t_0) = v_0,$$

respectively, existing on $[t_0, \infty)$, where $\delta \in C[J, R_+^n]$. Let, for $i = 1, 2,..., n$,

$$\Omega_i = [x, y \in R^n : \rho_i(t) - \epsilon_{2,i} < | x_i(t) - y_i(t)| < \rho_i(t)$$

and

$$r_i(t) < | x_i(t) - y_i(t)| < r_i(t) + \epsilon_{1,i}, t > t_0],$$

and, for each $t > t_0$, $x, y \in \Omega_i$,

$$| f_i(t, x) - f_{1,i}(t, y)| \leqslant g_i(t, | x - y |).$$

Then

$$v_0 \leqslant | x_0 - y_0 | \leqslant u_0$$

implies

$$\rho(t) \leqslant | x(t) - y(t)| \leqslant r(t), \qquad t \geqslant t_0,$$

$x(t)$, $y(t)$ being a solution and a δ-approximate solution of (2.7.1) and (2.7.3), respectively, existing on $[t_0, \infty)$.

The proof of this theorem can be constructed by following the respective arguments of Theorems 2.7.3, 2.7.1, and 1.5.1 with necessary modifications.

THEOREM 2.8.4. Let $\delta \in C[J, R_+^n]$, $g \in C[J \times R_+^n, R^n]$, and g possess the mixed quasi-monotone property. Suppose that $f, f_1 \in C[J \times R^n, R^n]$, and, for each $t \geqslant t_0$, $p = 1, 2,..., k$, $q = k + 1, k + 2,..., n$,

$$f_p(t, x) - f_{1,p}(t, y) \leqslant g_p(t, | x - y |), \qquad x_p \geqslant y_p,$$
$$f_q(t, x) - f_{1,q}(t, y) \geqslant g_q(t, | x - y |), \qquad x_q \geqslant y_q,$$

and $x(t)$, $y(t)$ are any solution and a δ-approximate solution of (2.7.1) and (2.7.3), respectively, on $[t_0, \infty)$.

(i) If $r(t)$ is the k max$(n - k)$ mini-solution of

$$u' = g(t, u) + \delta(t), \qquad u(t_0) = u_0$$

existing on $[t_0, \infty)$ such that

$$| x_{0,p} - y_{0,p} | \leqslant u_{0,p} ; \qquad | x_{0,q} - y_{0,q} | \geqslant u_{0,q} ,$$

then

$$| x_p(t) - y_p(t)| \leqslant r_p(t); \qquad | x_q(t) - y_q(t)| \geqslant r_q(t), \qquad t \geqslant t_0 .$$

(ii) If $v \in C[J, R_+^n]$, and, for $t > t_0$,

$$D_-v_p(t) > g_p(t, v(t)) + \delta_p(t),$$
$$D_-v_q(t) < g_q(t, v(t)) + \delta_q(t),$$

then, for $t \geqslant t_0$, we have

$$\begin{aligned} | x_p(t) - y_p(t)| &< v_p(t), \\ | x_q(t) - y_q(t)| &> v_q(t), \end{aligned} \tag{2.8.5}$$

whenever

$$| x_{0,p} - y_{0,p} | < v_p(t_0), \qquad | x_{0,q} - y_{0,q} | > v_q(t_0). \tag{2.8.6}$$

(iii) If $v \in C[J, R_+^n]$ satisfying

$$D_-v_p(t) \geqslant g_p(t, v(t)) + \delta_p(t),$$
$$D_-v_q(t) \leqslant g_q(t, v(t)) + \delta_q(t),$$

for $t > t_0$, then (2.8.6) implies (2.8.5) provided that, for each $\tau \in [t_0, \infty)$, $t \in [t_0, \tau]$, and for each $i = 1, 2,..., n$,

$$g_i(t, u) - g_i(t, \bar{u}) \geqslant -G(\tau + t_0 - t, u_i - \bar{u}_i),$$
$$u_i \geqslant \bar{u}_i , \qquad u_j = \bar{u}_j , \qquad i \neq j,$$

where $G \in C[J \times R_+, R_+]$, and $r(t) \equiv 0$ is the maximal solution of (2.7.10).

THEOREM 2.8.5. Let $g \in C[J \times R_+^n, R_+^n]$ and possess the quasi-monotone nondecreasing property. Let $r(t)$ be the maximal solution of

$$u' = g(t, u) + \delta(t), \qquad u(t_0) = u_0 ,$$

existing on $[t_0, \infty)$, where $\delta \in C[J, R_+^n]$. Assume that, for $t \in J$, $x, y \in R^n$,

$$| x - y + h[f(t, x) - f_1(t, y)]| \leqslant | x - y | + hg(t, | x - y |) + 0(h)$$

for all sufficiently small $h > 0$. Then,

$$|x_0 - y_0| \leqslant u_0$$

implies

$$|x(t) - y(t)| \leqslant r(t), \qquad t \geqslant t_0,$$

$x(t)$, $y(t)$ being a solution and a δ approximate solution of (2.7.1) and (2.7.3), respectively, existing on $[t_0, \infty)$.

2.9. Asymptotic equilibrium

We shall continue to consider the differential system (2.7.1).

DEFINITION 2.9.1. We shall say that the differential system (2.7.1) has *asymptotic equilibrium* if every solution of the system (2.7.1) tends to a finite limit vector ξ as $t \to \infty$ and to every constant vector ξ there is a solution $x(t)$ of (2.7.1) on $t_0 \leqslant t < \infty$ such that $\lim_{t\to\infty} x(t) = \xi$.

The following theorem gives sufficient conditions for the system (2.7.1) to have asymptotic equilibrium:

THEOREM 2.9.1. Let $f \in C[J \times R^n, R^n]$ and

$$\|f(t, x)\| \leqslant g(t, \|x\|), \qquad (t, x) \in J \times R^n, \tag{2.9.1}$$

where $g \in C[J \times R_+, R_+]$ and monotone nondecreasing in u for each $t \in J$. Assume that all solutions $u(t)$ of

$$u' = g(t, u), \qquad u(t_0) = u_0 > 0 \tag{2.9.2}$$

are bounded on $[t_0, \infty)$. Then the system (2.7.1) has asymptotic equilibrium.

Proof. Let $x(t)$ be any solution of (2.7.1). Then, it is easy to deduce from Theorem 2.7.1 that

$$\|x(t)\| \leqslant r(t), \qquad t \geqslant t_0, \tag{2.9.3}$$

where $r(t)$ is the maximal solution of (2.9.2) such that $\|x_0\| = u_0$.

Since, by assumption, every solution of (2.9.2) is bounded on $[t_0, \infty)$, it follows from (2.9.3) that every solution $x(t)$ of (2.7.1) is bounded on $[t_0, \infty)$.

Furthermore, for any $t > t_1 > t_0$, we have

$$\begin{aligned} \| x(t) - x(t_1)\| &\leqslant \int_{t_1}^{t} \| f(s, x(s))\| \, ds \\ &\leqslant \int_{t_1}^{t} g(s, \| x(s)\|) \, ds \\ &\leqslant \int_{t_1}^{t} g(s, r(s)) \, ds \\ &= r(t) - r(t_1), \end{aligned} \tag{2.9.4}$$

using (2.9.1) and (2.9.3) and the monotonic character of $g(t, u)$ in u. Since g is nonnegative, every solution $u(t)$ of (2.9.2) is nondecreasing in t, and hence the boundedness of all solutions of (2.9.2) shows that the maximal solution $r(t)$ tends to a limit as $t \to \infty$. This implies that, given an $\epsilon > 0$, we can choose a $t_1 > 0$ sufficiently large so that

$$0 \leqslant r(t) - r(t_1) < \epsilon \qquad \text{for all} \quad t > t_1 .$$

It then follows, because of (2.9.4), that

$$\| x(t) - x(t_1)\| < \epsilon \qquad \text{for all} \quad t > t_1 ,$$

which proves that

$$\lim_{t \to \infty} x(t) = \xi. \tag{2.9.5}$$

To prove that the system (2.7.1) has asymptotic equilibrium, it remains to be shown that, for every constant vector ξ such that $\| \xi \| \leqslant u_0$, there exists a solution $x(t)$ of (2.7.1) on $[t_0, \infty)$ such that (2.9.5) holds. For this purpose, let $x_n(t)$ be a solution of (2.7.1) such that

$$x_n(t_0 + n) = \xi \qquad (n = 1, 2, \ldots). \tag{2.9.6}$$

If $r_n(t)$ is the maximal solution of (2.9.2) with $r_n(t_0 + n) = \| \xi \|$, because of the nondecreasing character of every solution of (2.9.2), it follows that

$$\| \xi \| \leqslant r_n(t_0 + n) \leqslant r(t_0 + n).$$

We claim that

$$r_n(t) \leqslant r(t), \qquad t \geqslant t_0 + n. \tag{2.9.7}$$

It this were not true, let, for some $\sigma > t_0 + n$,

$$r_n(\sigma) > r(\sigma).$$

Then, by taking the larger of $r_n(t)$ and $r(t)$, we can construct a solution of (2.9.2) through $(t_0, \| \xi \|)$ whose value at σ is greater than that of the maximal solution $r(t)$, which is absurd. Hence, (2.9.7) is true. As before, for any $t_1 > t_0$ and $t > t_1$,

$$\| x_n(t) - x_n(t_1)\| \leqslant \int_{t_1}^{t} g(s, \| x_n(s)\|) \, ds.$$

Since $\| x_n(t) \| \leqslant r_n(t)$, $t \geqslant t_0 + n$, (2.9.7), together with the monotonicity of g, yields that

$$\| x_n(t) - x_n(t_1)\| \leqslant r(t) - r(t_1),$$

which assures that $x_n(t)$ tends to a limit vector $x_n(\infty)$ as $t \to \infty$ uniformly in n. Therefore, $\| f(t, x_n(t)) \|$ is bounded on every bounded t-interval uniformly in n. Since

$$x_n'(t) = f(t, x_n(t)),$$

the family $\{x_n(t)\}$ is equicontinuous on every bounded t-interval. Thus, there is a subsequence $\{x_{n_k}(t)\}$ that converges uniformly on every bounded t-interval as $k \to \infty$. As $\lim_{t\to\infty} x_n(t) \to x_n(\infty)$ for each n, this subsequence may be chosen such that the corresponding sequence of limits $\{x_{n_k}(\infty)\}$ also converges as $k \to \infty$. Now, $\{x_{n_k}(t)\}$ converges uniformly on $[t_0, \infty)$ to a continuous limit function $x(t)$ as $k \to \infty$. Evidently, $x(t)$ is a solution of (2.7.1), and therefore

$$\lim_{t\to\infty} x(t) = x(\infty).$$

Also, as $k \to \infty$,

$$\sup_{t_0 \leqslant t < \infty} \| x_{n_k}(t) - x(t)\| \to 0,$$

$$x_{n_k}(\infty) \to x(\infty),$$

and these two facts, together with (2.9.6), imply that $x(\infty) = \xi$. Now $x(t)$ is the desired solution, and this completes the proof of the theorem.

COROLLARY 2.9.1. If the function $g(t, u)$ in (2.9.1) is of the form

$$g(t, u) = \lambda(t)\phi(u),$$

where $\lambda(t) \geqslant 0$ is continuous for $t \in J$ and $\phi(u) \geqslant 0$ is continuous for $u \geqslant 0$, $\phi(0) = 0$, $\phi(u) > 0$, $u > 0$, and montonic nondecreasing in u, and if

$$\int_{t_0}^{\infty} \lambda(s) \, ds < \int_{u_0}^{\infty} du/\phi(u) \leqslant \infty, \tag{2.9.8}$$

the conclusion of Theorem 2.9.1 holds.

Theorem 2.9.1 has a corollary for the case that (2.7.1) is replaced by

$$x' = A(t)x + F(t, x), \tag{2.9.9}$$

where $A(t)$ is a continuous $n \times n$ matrix and $F \in C[J \times R^n, R^n]$. Let $X(t)$ be a fundamental matrix for

$$x' = A(t)x, \qquad X(t_0) = \text{unit matrix}, \tag{2.9.10}$$

so that the transformation

$$x = X(t)y$$

reduces (2.9.9) to

$$y' = X^{-1}(t)F(t, X(t)y). \tag{2.9.11}$$

Thus, an application of Theorem 2.9.1 to (2.9.11) gives

COROLLARY 2.9.2. Let $A(t)$ be a continuous matrix for $t \in J$ and $X(t)$ be a fundamental matrix for (2.9.10). Let $F \in C[J \times R^n, R^n]$, and, for $(t, y) \in J \times R^n$,

$$\| X^{-1}(t)F(t, X(t)y)\| \leqslant \lambda(t)\| y \|, \tag{2.9.12}$$

where $\lambda(t) \geqslant 0$ is continuous for $t \in J$, and

$$\int_{t_0}^{\infty} \lambda(s)\, ds < \infty.$$

Furthermore, let $x(t)$ be a solution of (2.9.9) on some t-interval to the right of t_0. Then $x(t)$ exists for all $t \geqslant t_0$,

$$\lim_{t\to\infty} X^{-1}(t)x(t) = \xi, \tag{2.9.13}$$

and, conversely, given a constant vector ξ, there is a solution $x(t)$ of (2.9.9) satisfying (2.9.13).

An interesting special case in which the hypotheses of Corollary 2.9.1 are satisfied is that of the linear homogeneous system (2.9.10), where

$$\int_{t_0}^{\infty} \| A(s)\|\, ds < \infty.$$

It is enough to take $\lambda(t) = \| A(t) \|$ and $\phi(u) = u$.

2.10. Asymptotic equivalence

Suppose we are given the following two differential systems:

$$x' = f_1(t, x), \qquad x(t_0) = x_0, \tag{2.10.1}$$

$$y' = f_2(t, y), \qquad y(t_0) = y_0, \tag{2.10.2}$$

where $f_1, f_2 \in C[J \times R^n, R^n]$. We shall first define asymptotic equivalence.

DEFINITION 2.10.1. The differential systems (2.10.1) and (2.10.2) are said to be *asymptotically equivalent* if, for every solution $y(t)$ of (2.10.2) [$x(t)$ of (2.10.1)], there is a solution $x(t)$ of (2.10.1) [$y(t)$ of (2.10.2)] such that

$$x(t) - y(t) \to 0 \quad \text{as} \quad t \to \infty.$$

THEOREM 2.10.1 Let $u(t)$ be a positive solution of

$$u' > g(t, u)$$

for $t \geqslant t_0$ such that $u(t) \to 0$ as $t \to \infty$, where $g \in C[J \times R_+, R]$. Suppose further that

$$\| x - y + h(f_1(t, x) - f_2(t, y))\| \leqslant \| x - y \| + hg(t, \| x - y \|) + 0(h) \quad (2.10.3)$$

for all sufficiently small h, $t \geqslant t_0$, and $\| x - y \| = u(t)$. Then, the systems (2.10.1) and (2.10.2) are asymptotically equivalent. If, in addition, one of the systems has asymptotic equilibrium, then the other system also has asymptotic equilibrium.

Proof. Let us first suppose that $y(t)$ is a solution of (2.10.2) defined for $t \geqslant t_0$. Let $x(t)$ be a solution of (2.10.1), defined on some t-interval to the right of t_0 such that

$$\| x(t_0) - y(t_0)\| \leqslant u(t_0).$$

Clearly, such a solution exists. Define

$$m(t) = \| x(t) - y(t)\|.$$

Then

$$m(t) \leqslant u(t),$$

as far as $x(t)$ exists. If this assertion is false, let t_1 be the greatest lower bound of numbers $t > t_0$, for which $m(t) \leqslant u(t)$ does not hold. Since $m(t)$ and $u(t)$ are continuous functions, we have, at $t = t_1$,

$$m(t_1) = u(t_1)$$

and

$$m(t_1 + h) > u(t_1 + h), \qquad h > 0.$$

This implies the inequality

$$D^+m(t_1) \geqslant u'(t_1) > g(t_1, u(t_1)). \quad (2.10.4)$$

In view of the condition (2.10.3), one also gets, at $t = t_1$,

$$D^+m(t_1) \leqslant g(t_1, m(t_1)),$$

which is a contradiction to (2.10.4). Hence,

$$\| x(t) - y(t)\| \leqslant u(t)$$

is true as far as $x(t)$ exists. Now, using Corollary 1.1.2, it follows that $x(t)$ exists for all $t \geqslant t_0$, since $y(t)$ and $u(t)$ are assumed to exist for $t \geqslant t_0$. Moreover, as $\lim_{t\to\infty} u(t) = 0$,

$$\lim_{t\to\infty} \| x(t) - y(t)\| = 0. \tag{2.10.5}$$

On the other hand, if $x(t)$ is a solution of (2.10.1) existing on $[t_0, \infty)$, arguing as before, we can conclude that there exists a solution $y(t)$ of (2.10.2) on $[t_0, \infty)$ such that (2.10.5) is satisfied. It therefore follows that the systems (2.10.1) and (2.10.2) are asymptotically equivalent.

If one of the systems has asymptotic equilibrium, the asymptotic equilibrium of the other system is a consequence of (2.10.5). The proof is complete.

The next theorem gives sufficient conditions for the asymptotic equivalence of the systems (2.9.9) and (2.9.10).

THEOREM 2.10.2. Let $A(t)$ be a continuous matrix for $t \in J$ and $F \in C[J \times R^n, R^n]$. Suppose that

$$\| F(t, x)\| \leqslant \lambda(t)\| x \|. \tag{2.10.6}$$

where $\lambda(t) \geqslant 0$ is continuous for $t \in J$, such that

$$\int_{t_0}^{\infty} \lambda(s)\, ds < \infty.$$

Assume that all the solutions of (2.9.10) are bounded as $t \to \infty$ and

$$\liminf_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} A(s)\, ds > -\infty. \tag{2.10.7}$$

Then, the systems (2.9.9) and (2.9.10) are asymptotically equivalent.

Proof. Let $Y(t)$ be a fundamental matrix of (2.9.10). Setting

$$Y(t)v(t) = x(t),$$

it is easy to verify that $x(t)$ is a solution of (2.9.9) if and only if $v(t)$ satisfies

$$v' = Y^{-1}(t)F(t, Y(t)v). \tag{2.10.8}$$

Using (2.10.6), (2.10.7), and the assumption that all the solutions of (2.9.10) are bounded, we get

$$\begin{aligned} \| Y^{-1}(t)F(t, Y(t)v)\| &\leqslant \| Y^{-1}(t)\| \| Y(t)\| \| v \|\lambda(t) \\ &\leqslant K\| v \|\lambda(t), \end{aligned}$$

where K is some constant. Hence, Corollary 2.9.1 implies that (2.10.8) has asymptotic equilibrium. Now, any solution $y(t)$ of (2.9.10) can be written as

$$y(t) = Y(t)\xi,$$

ξ being a constant column vector. Therefore,

$$x(t) - y(t) = Y(t)[v(t) - \xi],$$

and the desired result follows, since $Y(t)$ is bounded on (t_0, ∞).

The asymptotic equivalence of the systems (2.10.1) and (2.10.2) can also be considered on the basis of the variation of parameters formula for nonlinear systems developed in Theorem 2.6.3.

THEOREM 2.10.3. Assume that (i) $f_1, f_2 \in C[J \times R^n, R^n]$,

$$\frac{\partial f_1}{\partial x} \left(\frac{\partial f_2}{\partial y}\right)$$

exists and is continuous on $J \times R^n$; (ii) $\Phi_1(t, t_0, x_0)(\Phi_2(t, t_0, y_0))$ is the fundamental matrix solution of the variational system

$$\begin{aligned} z_1' &= \frac{\partial f_1(t, x(t, t_0, x_0))}{\partial x} z_1, \\ \left(z_2' \right.&= \left.\frac{\partial f_2(t, y(t, t_0, y_0))}{\partial y} z_2\right), \end{aligned} \tag{2.10.9}$$

such that $\Phi_1(t, t_0, x_0) = I$, $(\Phi_2(t_0, t_0, y_0) = I)$; (iii) for a given solution $y(t)$ of (2.10.2) [$x(t)$ of (2.10.1)], existing on $[t_0, \infty)$,

$$\int_t^\infty \Phi_1(t, s, y(s))[f_2(s, y(s)) - f_1(s, y(s))]\, ds \to 0 \quad \text{as} \quad t \to \infty,$$

$$\left(\int_t^\infty \Phi_2(t, s, x(s))[f_1(s, x(s)) - f_2(s, x(s))]\, ds \to 0 \quad \text{as} \quad t \to \infty\right).$$

Then, there exists a solution $x(t)$ of (2.10.1) [$y(t)$ of (2.10.2)] on $[t_0, \infty)$ satisfying the relation

$$\lim_{t\to\infty} x(t) - y(t) = 0. \tag{2.10.10}$$

Proof. Let $y(t) = y(t, t_0, y_0)$ be a given solution of (2.10.2) existing on $[t_0, \infty)$. Define a function $x(t)$ by the relation

$$x(t) = y(t) + \int_t^\infty \Phi_1(t, s, y(s))[f_2(s, y(s)) - f_1(s, y(s))]\, ds. \tag{2.10.11}$$

Since the integral converges by assumption (iii), it follows that $x(t)$ is well defined, and, consequently, (2.10.10) is satisfied. It therefore suffices to prove that $x(t)$ is a solution of (2.10.1). For this purpose, we observe, as in Theorem 2.6.3, that

$$\frac{dx(t, s, y(s))}{ds} = \Phi_1(t, s, y(s))[f_2(s, y(s)) - f_1(s, y(s))]. \tag{2.10.12}$$

Here use is made of the relation (2.5.11) and the fact that

$$\frac{\partial x(t, s, y(s))}{\partial x_0} = \Phi_1(t, s, y(s)).$$

The relations (2.10.11) and (2.10.12) yield

$$\begin{aligned} x(t) &= y(t) + \int_t^\infty \frac{dx(t, s, y(s))}{ds}\, ds \\ &= y(t) + \lim_{T\to\infty} \int_t^T \frac{dx(t, s, y(s))}{ds}\, ds \\ &= y(t) + \lim_{T\to\infty} x(t, T, y(T)) - y(t) \\ &= \lim_{T\to\infty} x(t, T, y(T)). \end{aligned}$$

The continuity of $f_1(t, x)$ now assures that

$$\begin{aligned} \lim_{T\to\infty} f_1(t, x(t, T, y(T))) &= f_1(t, \lim_{T\to\infty} x(t, T, y(T))) \\ &= f_1(t, x(t)). \end{aligned} \tag{2.10.13}$$

Moreover, we have

$$\frac{df_1(t, x(t, s, y(s)))}{ds} = \frac{\partial f_1(t, x(t, s, y(s)))}{\partial x} \cdot \frac{dx(t, s, y(s))}{ds}. \tag{2.10.14}$$

Let us differentiate (2.10.11), recalling that $y(t)$, $\Phi_1(t, t_0, x_0)$ are the solutions of (2.10.2) and (2.10.9), respectively, and using (2.10.12) to obtain

$$\begin{aligned} x'(t) &= f_2(t, y(t)) - \Phi_1(t, t, y(t))[f_2(t, y(t)) - f_1(t, y(t))] \\ &\quad + \int_t^\infty \frac{\partial \Phi_1(t, s, y(s))}{\partial t} [f_2(s, y(s)) - f_1(s, y(s))]\, ds \\ &= f_1(t, y(t)) + \int_t^\infty \frac{\partial f_1(t, x(t, s, y(s))}{\partial x} \cdot \frac{dx(t, s, y(s))}{ds}\, ds. \end{aligned}$$

This reduces to, in view of (2.10.14),

$$\begin{aligned} x'(t) &= f_1(t, y(t)) + \int_t^\infty \frac{df_1(t, x(t, s, y(s)))}{ds}\, ds \\ &= f_1(t, y(t)) + \lim_{T\to\infty} \int_t^T \frac{df_1(t, x(t, s, y(s)))}{ds}\, ds \\ &= f_1(t, y(t)) + \lim_{T\to\infty} f_1(t, x(t, T, y(T))) - f_1(t, y(t)) \\ &= \lim_{T\to\infty} f_1(t, x(t, T, y(T))). \end{aligned}$$

The relation (2.10.13) implies that $x(t)$ is a solution of (2.10.1) with

$$x_0 = y_0 + \int_{t_0}^\infty \Phi_1(t_0, s, y(s))[f_2(s, y(s)) - f_1(s, y(s))]\, ds.$$

On the other hand, if $x(t)$ is a solution of (2.10.1) existing on $[t_0, \infty)$, we can show exactly in a similar way that there exists a solution $y(t)$ of (2.10.2) on $[t_0, \infty)$ such that (2.10.10) holds. Thus the theorem is established.

2.11. A topological principle

This topological principle is concerned with the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geq 0, \tag{2.11.1}$$

where $f \in C[E, R^n]$, E being an open (t, x)-set in R^{n+1}. Let E_0 be an open subset of E, ∂E_0 the boundary, and $\bar{E}_0$ the closure of E_0.

DEFINITION 2.11.1. A point $(t_0, x_0) \in E \cap \partial E_0$ is said to be an *egress point* of E_0 with respect to the system (2.11.1) if, for every solution $x(t)$

of (2.11.1), there is an $\epsilon > 0$ such that $(t, x(t)) \in E_0$ for $t_0 - \epsilon \leqslant t \leqslant t_0$. An egress point (t_0, x_0) of E_0 is called a *strict egress point* of E_0 if $(t, x(t)) \notin \bar{E}_0$ for $t_0 < t \leqslant t_0 + \epsilon$, for a small $\epsilon > 0$.

The set of all points of egress (strict egress) is denoted by $S(S^*)$. It is clear that $S^* \subset S$.

DEFINITION 2.11.2. If $A \subset B$ are any two sets of a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. When there exists a retraction of B onto A, A is called a *retract* of B.

The following examples would suffice to illustrate the concept of retraction.

Example 1. Let $B = [x \in R^n : \| x \| \leqslant \alpha]$ and $A = [x \in R^n : \| x \| = \alpha]$. Then A is not a retract of B. For, if there exists a retraction $\pi : B \to A$, then there exists a continuous map of B into itself, $x \to -\pi(x)$, without fixed points. This contradicts the Brouwer's fixed point theorem.

Example 2. Let

$$C = [(x, u) \in R^{n+m} : \| x \| = \alpha,\ u \text{ arbitrary}],$$

$$B = [(x, u_0) \in R^{n+m} : \| x \| \leqslant \alpha,\ u_0 \text{ fixed}],$$

$$A = B \cap C = [(x, u_0) \in R^{n+m} : \| x \| = \alpha,\ u_0 \text{ fixed}].$$

From example 1, it can be seen that A is not a retract of B, whereas it is a retract of C, because we can choose a retraction $\pi(x, u) = (x, u_0)$.

THEOREM 2.11.1. Let $f \in C[E, R^n]$, where E is an open (t, x)-set in R^{n+1}. Assume that, through every point of E, there passes a unique solution of the system (2.11.1) and that the solutions depend continuously on initial values. Let E_0 be an open subset of E. Suppose that all egress points of E_0 are strict egress points, i.e., $S = S^*$. Let Z be a nonempty subset of $E_0 \cup S$ such that $Z \cap S$ is a retract of S, but is not a retract of Z. Then, there exists at least one point $(t_0, x_0) \in Z \cap E_0$ such that the solution arc $(t, x(t))$ of (2.11.1) remains in E_0 on its maximal interval of existence to the right of t_0.

Proof. Suppose that the conclusion of the theorem is not true. Then, for every $(t_0, x_0) \in Z - S$, there exists a $t_1 = t_1(t_0, x_0)$, $t_1 > t_0$ such that the solution $x(t) = x(t, t_0, x_0)$ of (2.11.1) exists on $[t_0, t_1]$, $(t, x(t)) \in E_0$ for $[t_0, t_1)$ and $(t_1, x(t_1)) \in S$.

Define a map $\pi_0 : Z \to S$ such that

(i) $\pi_0(t_0, x_0) = (t_1, x(t_1))$ if $(t_0, x_0) \in Z - S$;

(ii) $\pi_0(t_0, x_0) = (t_0, x_0)$ if $(t_0, x_0) \in Z \cap S$.

We shall show that π_0 is continuous because of the assumption $S = S^*$ and the continuous dependence of the solutions on the initial values. Let $(t_0, x_0) \in Z \cap E_0$ and (t^*, x^*) be sufficiently near to (t_0, x_0). Then the solution $x(t, t^*, x^*)$ exists on $[t^*, t_1 + \epsilon]$ for some small $\epsilon > 0$ and

(i) $(t, x(t, t^*, x^*)) \in E_0$, $[t^*, t_1 - \epsilon]$;

(ii) $(t, x(t, t^*, x^*)) \notin \bar{E}_0$, $t = t_1 + \epsilon$.

This implies that, at $t_1^* = t_1^*(t^*, x^*)$, $(t_1^*, x(t_1^*, t^*, x^*)) \in S$ and $| t_1^* - t_1 | < \epsilon$, which shows that $(t_1^*, x(t_1^*, t^*, x^*))$ is a continuous function of (t^*, x^*). This proves that π_0 is continuous at (t_0, x_0). A similar reasoning holds if $(t_0, x_0) \in Z \cap S$.

Let π be a retraction of S onto $Z \cap S$. It then follows that the composite map $\pi\pi_0 : Z \to Z \cap S$ is a retraction. This contradicts the assumption that $Z \cap S$ is not a retract of Z. The theorem is proved.

To give an idea of the interplay of the conditions in Theorem 2.11.1, let us consider (2.11.1), where $E = J \times R$, $E_0 = [(t, x) : t \in J, | x | < b]$. The boundary ∂E_0 consists of the half-lines $x = \pm b$. The assumption that $f(t, b) > 0$, $f(t, -b) < 0$ guarantees that $S = S^* = \partial E_0$. The set Z can be chosen as $Z = [(t, x) : t = t_0, | x | < b]$ and $Z \cap S$ as the set of two points $(t_0, \pm b)$. Then $Z \cap S$ is a retract of S but not of Z. Theorem 2.11.1 now shows that there exists at least one point (t_0, x_0), $| x_0 | < b$, such that a solution of (2.11.1) exists and satisfies $| x(t) | < b$ for $t \geq t_0$.

Given a differential system, the choice of the set E_0, for which Theorem 2.11.1 can be applied successfully, may be rather difficult. However, in some cases, it is possible to overcome this difficulty.

Let $u \in C[E, R^n]$ and $x(t)$ be a solution of (2.11.1). The function $u(t, x)$ is said to possess a trajectory derivative $u'(t, x)$ at the point (t_0, x_0) along the solution $x(t)$ of (2.11.1) if $u(t, x(t))$ has a derivative at $t = t_0$, in which case

$$u'(t_0, x_0) = [u(t, x(t))]'_{t=t_0}.$$

If $u(t, x)$ is continuously differentiable with respect to (t, x), the trajectory derivative $u'(t, x)$ exists and is equal to

$$\frac{\partial u(t, x)}{\partial t} + \frac{\partial u(t, x)}{\partial x} \cdot f(t, x),$$

where the centered dot denotes the usual scalar product of vectors. The following theorem provides a suitable choice of E_0 and the set of egress points in terms of certain functions.

THEOREM 2.11.2. Let $f \in C[E, R^n]$, $u \in C[E, R^p]$, and $v \in C[E, R^q]$. Let

(i) $E_0 = [(t, x) : u_j(t, x) < 0$ and $v_k(t, x) < 0, 1 \leqslant j \leqslant p, 1 \leqslant k \leqslant q]$;

(ii) $L_\alpha = [(t, x) : u_\alpha(t, x) = 0$ and $u_j(t, x) \leqslant 0, v_k(t, x) \leqslant 0, 1 \leqslant j \leqslant p, 1 \leqslant k \leqslant q]$;

(iii) $M_\beta = [(t, x) : v_\beta(t, x) = 0$ and $u_j(t, x) \leqslant 0, v_k(t, x) \leqslant 0, 1 \leqslant j \leqslant p, 1 \leqslant k \leqslant q]$.

Assume further that the trajectory derivatives $u'_\alpha(t, x)$, $v'_\beta(t, x)$ exist on L_α, M_β and satisfy

$$u'_\alpha(t, x) > 0, \qquad (t, x) \in L_\alpha, \tag{2.11.2}$$

$$v'_\beta(t, x) < 0, \qquad (t, x) \in M_\beta, \tag{2.11.3}$$

respectively, along all solutions through (t, x). Then

$$S = S^* = \bigcup_{\alpha=1}^{p} L_\alpha - \bigcup_{\beta=1}^{q} M_\beta .$$

Proof. We shall first show that $S \cap M_\beta$ is empty. For, if $(t_0, x_0) \in M_\beta$, and $x(t)$ is a solution of (2.11.1), from (2.11.3) we have

$$v_\beta(t, x(t)) > 0, \qquad [t_0 - \epsilon, t_0) \qquad \text{for small} \quad \epsilon > 0,$$

which shows that $(t, x(t)) \notin E_0$ because of (i). This means that (t_0, x_0) is not a point of egress. Since

$$\partial E_0 \cap E \subset \left(\bigcup_{\alpha=1}^{p} L_\alpha\right) \cup \left(\bigcup_{\beta=1}^{q} M_\beta\right),$$

it follows that

$$S^* \subset S \subset (\partial E_0 \cap E) - \bigcup_{\beta=1}^{q} M_\beta \subset \bigcup_{\alpha=1}^{p} L_\alpha - \bigcup_{\beta=1}^{q} M_\beta . \tag{2.11.4}$$

On the other hand, if

$$(t_0, x_0) \in \bigcup_{\alpha=1}^{p} L_\alpha - \bigcup_{\beta=1}^{q} M_\beta ,$$

then, from (ii) and (iii), we get $u_j(t_0, x_0) \leqslant 0$ and $v_k(t_0, x_0) < 0$, $j = 1,..., p$, $k = 1,..., q$. The assumption (2.11.2) yields that there is an $\epsilon > 0$ such that

$$u_\alpha(t, x(t)) < 0, \qquad [t_0 - \epsilon, t_0),$$

$$u_\alpha(t, x(t)) > 0, \qquad (t_0, t_0 + \epsilon]$$

if $(t_0, x_0) \in L_\alpha$;

$$u_j(t, x(t)) < 0, \qquad [t_0 - \epsilon, t_0 + \epsilon]$$

if $(t_0, x_0) \notin L_j$; and

$$v_k(t, x(t)) < 0, \qquad [t_0 - \epsilon, t_0 + \epsilon] \qquad \text{for all} \quad k.$$

Hence, $(t_0, x_0) \in S^*$, and

$$\bigcup_{\alpha=1}^{p} L_\alpha - \bigcup_{\beta=1}^{q} M_\beta \subset S^*.$$

This, together with (2.11.4), establishes the theorem.

2.12. Applications of topological principle

Consider the two differential systems

$$x' = f_1(t, x), \tag{2.12.1}$$

$$y' = f_2(t, y), \tag{2.12.2}$$

where $f_1, f_2 \in C[J \times R^n, R^n]$. Let $g \in C[J \times R_+, R]$ and $u(t)$ be a positive solution of the differential inequality

$$u' < g(t, u). \tag{2.12.3}$$

Given that $y(t)$ is a solution of (2.12.2) for $t \geqslant t_0$, we shall show that there is a solution $x(t)$ of (2.12.1) such that

$$\| x(t) - y(t) \| < u(t), \qquad t \geqslant t_0. \tag{2.12.4}$$

THEOREM 2.12.1. Let $f_1, f_2 \in C[J \times R^n, R^n]$ and $g \in C[J \times R_+, R]$. Assume that the systems (2.12.1) and (2.12.2) possess unique solutions through every point and that the solutions depend continuously on the

initial values. Let $y(t)$ be a solution of (2.12.2) and $u(t)$ a positive solution of (2.12.3), for $t \geqslant t_0$. Suppose further that

$$\begin{aligned}\| x - y(t) + h(f_1(t, x) - f_2(t, y(t)))\| \\ \geqslant \| x - y(t)\| + hg(t, \| x - y(t)\|) \qquad (2.12.5)\end{aligned}$$

for all sufficiently small h and $\| x - y(t) \| = u(t)$. Then, if $\tau > t_0$ is given, there exists a solution $x(t)$ of (2.12.1) defined for $t \geqslant \tau$ satisfying (2.12.4) for $t \geqslant \tau$.

Proof. We wish to apply Theorems 2.11.1 and 2.11.2 to deduce the result. Defining

$$\begin{aligned} E_0 &= [(t, x) : \| x - y(t)\| < u(t), t > t_0], \\ u(t, x) &= \| x - y(t)\| - u(t), \\ v(t, x) &= t_0 - t, \end{aligned}$$

it follows that

$$\begin{aligned} E_0 &= [(t, x) : u(t, x) < 0, v(t, x) < 0], \\ L &= [(t, x) : u(t, x) = 0, v(t, x) \leqslant 0], \\ M &= [(t, x) : u(t, x) \leqslant 0, v(t, x) = 0]. \end{aligned}$$

If $\alpha(t) = \| x(t) - y(t) \|$, where $x(t)$ is a solution of (2.12.1) such that, for some $t = t_1 \geqslant t_0$, $\alpha(t_1) = u(t_1)$, the condition (2.12.5) yields

$$\begin{aligned} \alpha(t_1 + h) &= \| x(t_1 + h) - y(t_1 + h)\| \\ &= \| x(t_1) + hf_1(t_1, x(t_1)) + \epsilon_1(h) - y(t_1) - hf_2(t_1, y(t_1)) - \epsilon_2(h)\| \\ &\geqslant \alpha(t_1) + hg(t_1, \alpha(t_1)) + \epsilon(h), \end{aligned}$$

where $\epsilon_1(h)/h$, $\epsilon_2(h)/h$, and $\epsilon(h)/h$ all tend to zero as $h \to 0$. This implies the inequality

$$\alpha'(t_1) \geqslant g(t_1, \alpha(t_1)) = g(t_1, u(t_1)). \qquad (2.12.6)$$

Using the inequalities (2.12.3) and (2.12.6), we obtain

$$\begin{aligned} u'(t_1, x_1) &= \alpha'(t_1) - u'(t_1) \\ &\geqslant g(t_1, u(t_1)) - u'(t_1) \\ &> 0, \qquad (t_1, x_1) \in L, \end{aligned}$$

where $x_1 = x(t_1)$. Moreover,

$$v'(t_1, x_1) = -1, \qquad (t_1, x_1) \in M.$$

Thus, in view of Theorem 2.11.2, we have

$$S = S^* = L - M.$$

Let $\tau > t_0$ be given. Since

$$S = [(t, x) : t > t_0\,, \| x - y(t)\| = u(t)],$$

defining

$$Z = [(t, x) : t = \tau, \| x - y(t)\| \leqslant u(\tau)],$$

one sees that

$$Z \cap S = [(t, x) : t = \tau, \| x - y(\tau)\| = u(\tau)].$$

Observing that Z is a closed ball in R^n and $Z \cap S$ is the boundary of the ball Z in R^n, it is clear that $Z \cap S$ is not a retract of Z. However, the mapping $\pi : S \to S \cap Z$ given by

$$\pi(t, x) = (t^*, x^*), \qquad \text{with} \quad t^* = \tau,$$

$$x^* = y(\tau) + (x - y(t))[u(\tau)/u(t)],$$

is a retraction. Consequently, we conclude from Theorem 2.11.1 that there exists at least one point $(\tau, x_0) \in Z - S$ such that the solution arc $(t, x(t, \tau, x_0))$ of (2.12.1) remains in E_0 on its maximal interval of existence. Since $u(t)$ and $y(t)$ exist for all $t \geqslant \tau \geqslant t_0$, it follows that the maximal interval of existence of $x(t)$ is $[\tau, \infty)$. Hence (2.12.4) holds, and the proof is complete.

REMARK 2.12.1. If, for every solution $y(t)$ of (2.12.2) [$x(t)$ of (2.12.1)], there exists a t_0 and $g(t, u)$ such that (2.12.3) is satisfied by some positive function $u(t)$ that tends to zero as $t \to \infty$ and (2.12.5) is satisfied for $\| x - y(t) \| = u(t)$ ($\| x(t) - y \| = u(t)$), then the systems (2.12.1) and (2.12.2) are asymptotically equivalent.

2.13. Stability criteria

We consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0\,, \tag{2.13.1}$$

where $f \in C[J \times S_\rho\,, R^n]$, S_ρ being the set

$$S_\rho = [x \in R^n : \| x \| < \rho].$$

Assume that $f(t, 0) \equiv 0$, so that (2.13.1) admits the trivial solution. Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.13.1) through (t_0, x_0).

DEFINITION 2.13.1. The trivial solution of (2.13.1) is said to be (i) *stable* if, for every $\epsilon > 0$ and $t_0 \in J$, there exists a $\delta > 0$ such that $\| x_0 \| < \delta$ implies $\| x(t) \| < \epsilon$, $t \geqslant t_0$; (ii) *asymptotically stable* if it is stable and if there exists a $\delta_0 > 0$ such that $\| x_0 \| < \delta_0$ implies $x(t) \to 0$ as $t \to \infty$.

THEOREM 2.13.1. Let $g \in C[J \times R_+, R]$ and $g(t, 0) \equiv 0$. Assume that the function $f(t, x)$ satisfies

$$\| x + hf(t, x)\| \leqslant \| x \| + hg(t, \| x \|) + 0(h) \tag{2.13.2}$$

for $(t, x) \in J \times S_\rho$ and for all sufficiently small $h > 0$. Then, the stability or asymptotic stability of the trivial solution of

$$u' = g(t, u), \qquad u(t_0) = u_0 \tag{2.13.3}$$

implies likewise the stability or asymptotic stability of the trivial solution of the system (2.13.1).

Proof. Let the solution $u = 0$ of (2.13.3) be stable. Then, given $0 < \epsilon < \rho$, and $t_0 \in J$, there exists a $\delta > 0$ with the property that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon$, $t \geqslant t_0$. It is easy to claim that, with these ϵ and δ, the trivial solution of (2.13.1) is stable. If this were false, there would exist a solution $x(t)$ of (2.13.1) and a $t_1 > t_0$ such that

$$\| x(t_1)\| = \epsilon, \qquad \| x(t)\| \leqslant \epsilon, \qquad t_0 \leqslant t \leqslant t_1 .$$

For $t \in [t_0, t_1]$, using the condition (2.13.2), it follows that

$$D^+ m(t) \leqslant g(t, m(t)), \tag{2.13.4}$$

where $m(t) = \| x(t) \|$, and hence, by Theorem 1.4.1, choosing $\| x_0 \| = u_0$, we obtain

$$\| x(t)\| \leqslant r(t, t_0, \| x_0 \|), \qquad t \in [t_0, t_1], \tag{2.13.5}$$

where $r(t, t_0, \| x_0 \|)$ is the maximal solution of (2.13.3). At $t = t_1$, we therefore arrive at the following contradiction:

$$\epsilon = \| x(t_1)\| \leqslant r(t_1, t_0, \| x_0 \|) < \epsilon,$$

thus justifying our claim.

Suppose that the solution $u = 0$ is asymptotically stable. Since this implies, by definition, stability of $u = 0$, the stability of the trivial solution of (2.13.1) is a consequence of the foregoing argument. This means that the inequality (2.13.4) holds for all $t \geqslant t_0$, and hence (2.13.5) is valid for $t \geqslant t_0$. It is now clear, by hypothesis, that, if $\| x_0 \| < \delta_0$, $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

Let us demonstrate the significance and practicability of the assumptions of Theorem 2.13.1 by an example.

Let $f(t, x) = Ax$, where A is an $n \times n$ constant matrix. Since

$$\| x + hAx \| \leqslant \| I + hA \| \| x \|,$$

it follows that

$$\| x + hAx \| - \| x \| \leqslant [\| I + hA \| - 1] \| x \|.$$

Thus, defining the logarithmic norm

$$\mu(A) = \lim_{h\to 0^+} \frac{[\| I + hA \| - 1]}{h}, \tag{2.13.6}$$

We see that

$$\| x + hAx \| \leqslant \| x \| [1 + h\mu(A) + \epsilon(h)],$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. The function $g(t, u)$ is therefore given by

$$g(t, u) = \mu(A)u.$$

Clearly, $g(t, 0) \equiv 0$, and the general solution of (2.13.3) is

$$u(t, t_0, u_0) = u_0 \exp[\mu(A)(t - t_0)].$$

Thus, the trivial solution $u = 0$ of (2.13.3) is stable if $\mu(A) \leqslant 0$ and asymptotically stable if $\mu(A) < 0$. Hence, the stability or asymptotic stability of the trivial solution of (2.13.1) follows from Theorem 2.13.1.

From the definition (2.13.6), it is easily seen that

$$\mu(\alpha A) = \alpha\mu(A), \qquad \alpha \geqslant 0, \tag{2.13.7}$$

$$| \mu(A) | \leqslant \| A \|, \tag{2.13.8}$$

$$\mu(A + B) \leqslant \mu(A) + \mu(B), \tag{2.13.9}$$

and, from (2.13.8) and (2.13.9),

$$| \mu(A) - \mu(B) | \leqslant \| A - B \|. \tag{2.13.10}$$

The value of $\mu(A)$ depends on the particular norm used for vectors and matrices. For example, if $\| x \|$ represents the Euclidean norm, $\mu(A)$ is the largest eigenvalue of $\frac{1}{2}[A + A^*]$, A^* being the transpose of A, whereas the corresponding matrix norm $\| A \|$ is the square root of the largest eigenvalue of A^*A. On the other hand, if $\| x \| = \sum_{i=1}^{n} | x_i |$, and $\| A \| = \sup_k \sum_{i=1}^{n} | a_{ik} |$, then

$$\mu(A) = \sup_k \left[\operatorname{Re} a_{kk} + \sum_{i, i \neq k}^{n} | a_{ik} | \right].$$

We further remark that every eigenvalue of A has real part less than or equal to $\mu(A)$. For, if λ is an eigenvalue of A, and x a corresponding eigenvector of norm 1, then

$$\|(I + hA)x\| - \| x \| = | 1 + h\lambda | - 1 \sim h \operatorname{Re} \lambda \qquad \text{for} \quad h \to 0^+.$$

On the other hand,

$$\|(I + hA)x\| - \| x \| \leqslant \| I + hA \| - 1 \sim h\mu(A) \qquad \text{for} \quad h \to 0^+.$$

Therefore, $\operatorname{Re} \lambda \leqslant \mu(A)$.

Let us now take $f(t, x) = A(t)x$, where $A(t)$ is a continuous $n \times n$ matrix on J. In this case, $g(t, u) = \mu[A(t)]u$. We observe that $\mu[A(t)]$ is continuous on J, by virtue of the inequality (2.13.10) and the continuity of $A(t)$. The general solution of (2.13.3) is of the form

$$u(t, t_0, u_0) = u_0 \exp \left[\int_{t_0}^{t} \mu[A(s)]\, ds \right],$$

and, hence, the trivial solution $u = 0$ of (2.13.3) is stable if

$$\limsup_{t \to \infty} \int_{t_0}^{t} \mu[A(s)]\, ds < \infty$$

and is asymptotically stable if

$$\limsup_{t \to \infty} \int_{t_0}^{t} \mu[A(s)]\, ds = -\infty.$$

Therefore, the corresponding stability properties of the trivial solution of (2.13.1) follow from Theorem 2.13.1.

THEOREM 2.13.2. Assume that (i) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $f_x(t, x)$ exists and is continuous on $J \times S_\rho$; (ii) $\mu[f_x(t, 0)]$ satisfies

$$\limsup_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \mu[f_x(s, 0)]\, ds = \alpha < 0. \tag{2.13.11}$$

Then, the trivial solution of (2.13.1) is asymptotically stable.

Proof. Since $f(t, 0) \equiv 0$, given $\epsilon > 0$, it is possible to find a $\delta(\epsilon) > 0$ such that

$$f(t, x) = f_x(t, 0)x + F(t, x), \tag{2.13.12}$$

where

$$\| F(t, x)\| \leqslant \epsilon \| x \| \qquad \text{if} \quad \| x \| \leqslant \delta \tag{2.13.13}$$

uniformly in t.

Let $\epsilon > 0$ and $t_0 \in J$ be given. By the condition (2.13.11), it follows that we have, for large $t > t_0$,

$$\int_{t_0}^{t} \mu[f_x(s, 0)]\, ds \leqslant \frac{\alpha}{2}(t - t_0),$$

and, if ϵ is small enough,

$$\lim_{t \to \infty} \exp[\epsilon(t - t_0) + \int_{t_0}^{t} \mu[f_x(s, 0)]\, ds] = 0. \tag{2.13.14}$$

Thus,

$$K = \max_{t_0 \leqslant t < \infty} \exp[\epsilon(t - t_0) + \int_{t_0}^{t} \mu[f_x(s, 0)]\, ds]$$

exists, and we choose $K \geqslant 1$ and δ_1 such that

$$K\delta_1 \leqslant \delta(\epsilon). \tag{2.13.15}$$

Then, if $\| x_0 \| < \delta_1$, we claim that $\| x(t) \| < \delta$, $t \geqslant t_0$, where $x(t) = x(t, t_0, x_0)$ is any solution of (2.13.1). If this were false, there exists a $t_1 > t_0$ with the property that

$$\| x(t_1)\| = \delta, \qquad \| x(t)\| \leqslant \delta, \qquad t_0 \leqslant t \leqslant t_1. \tag{2.13.16}$$

Defining $m(t) = \| x(t) \|$, we observe that

$$\| x(t) + hf(t, x(t))\| \leqslant \| I + hf_x(t, 0)\| \| x(t)\| + h\| F(t, x(t))\|,$$

for $t \in [t_0, t_1]$, because of (2.13.2), and hence

$$m'_+(t) \leqslant \mu[f_x(t, 0)]m(t) + \| F(t, x(t))\|,$$

which, in view of the relations (2.13.16) and (2.13.13), yields

$$m'_+(t) \leqslant [\mu[f_x(t, 0)] + \epsilon]m(t), \qquad t \in [t_0, t_1]. \tag{2.13.17}$$

Theorem 1.4.1 then implies, choosing $u_0 = m(t_0)$, that, for $t \in [t_0, t_1]$,

$$m(t) \leqslant m(t_0) \exp \left[\epsilon(t - t_0) + \int_{t_0}^{t} \mu[f_x(s, 0)]\, ds\right], \tag{2.13.18}$$

and we are led to an absurdity:

$$\delta = m(t_1) = m(t_0) \exp\left[\epsilon(t_1 - t_0) + \int_{t_0}^{t_1} \mu[f_x(s, 0)]\, ds\right]$$
$$< K\delta_1 \leqslant \delta,$$

because of relations (2.13.15), (2.13.16), and the fact that $\| x_0 \| < \delta_1$. This proves that, whenever $\| x_0 \| < \delta_1$, we have $\| x(t) \| < \delta$, $t \geqslant t_0$, and therefore the inequality (2.13.17) is true for all $t \geqslant t_0$. Consequently, (2.13.18) holds for all $t \geqslant t_0$. It now follows from (2.13.14) and (2.13.18) that $\lim_{t\to\infty} x(t) = 0$, if ϵ is small enough, which establishes the stated result.

THEOREM 2.13.3. Assume that the hypothesis of Theorem 2.13.1 holds except that the inequality (2.13.2) is replaced by

$$x \cdot f(t, x) \leqslant \| x \| g(t, \| x \|), \qquad (t, x) \in J \times S_\rho . \tag{2.13.19}$$

Then, the conclusion of Theorem 2.13.1 remains valid.

Proof. We proceed as in Theorem 2.13.1 to get

$$\| x(t_1)\| = \epsilon, \qquad \| x(t)\| \leqslant \epsilon, \qquad t_0 \leqslant t \leqslant t_1 .$$

Now, using (2.13.19) and setting $m(t) = \| x(t) \|$, we obtain, for $t \in [t_0, t_1]$, the inequality

$$m(t)m'(t) \leqslant m(t)\, g(t, m(t)). \tag{2.13.20}$$

Choose $\| x_0 \| = u_0$. We wish to prove the relation (2.13.5). For this purpose, it is enough to show that

$$m(t) < u(t, \epsilon), \qquad t \in [t_0, t_1],$$

where $u(t, \epsilon)$ is a solution of

$$u' = g(t, u) + \epsilon, \qquad u(t_0) = u_0 + \epsilon, \tag{2.13.21}$$

ϵ being sufficiently small positive quantity.

Assuming the contrary and following the proof of Theorem 1.2.1, we arrive at a t_2, $t_0 < t_2 \leqslant t_1$ such that

$$m(t_2) = u(t_2, \epsilon), \qquad m'(t_2) \geqslant u'(t_2, \epsilon). \tag{2.13.22}$$

By assumption on $g(t, u)$, we see that $u(t_2, \epsilon) > 0$, and therefore $m(t_2) > 0$. Thus, the relations (2.13.20), (2.13.21), and (2.13.22) lead to the contradiction

$$g(t_2, u(t_2, \epsilon)) + \epsilon \leqslant g(t_2, m(t_2)),$$

proving $m(t) < u(t, \epsilon)$, $t \in [t_0, t_1]$, which implies (2.13.5) because of the fact that $\lim_{\epsilon \to 0} u(t, \epsilon) = r(t, t_0, \| x_0 \|)$ uniformly on $[t_0, t_1]$.

The rest of the proof is the same as in Theorem 2.13.1.

2.14. Asymptotic behavior

We shall present here several results on the asymptotic behavior of solutions of differential systems.

THEOREM 2.14.1. Assume that $F \in C[J \times R^n, R^n]$, and, for sufficiently small $h > 0$,

$$\| x - y + h(F(t, x) - F(t, y))\| \leqslant \| x - y \| + hg(t, \| x - y \|) + 0(h), \qquad (2.14.1)$$

where $g \in C[J \times R_+, R]$. If every solution $u(t)$ of

$$u' = g(t, u) + \| F(t, 0)\|, \qquad u(t_0) = u_0 > 0 \qquad (2.14.2)$$

tends to zero as $t \to \infty$, then every solution $x(t)$ of

$$x' = F(t, x), \qquad x(t_0) = x_0 \qquad (2.14.3)$$

tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be any solution of (2.14.3) such that $\| x_0 \| \leqslant u_0$. Define

$$m(t) = \| x(t)\|.$$

Then, for small $h > 0$, we have

$$\begin{aligned} m(t + h) &= \| x(t) + hF(t, x(t)) + \epsilon(h)\| \\ &\leqslant \| x(t) + h[F(t, x(t)) - F(t, 0)]\| \\ &\quad + h\| F(t, 0)\| + \| \epsilon(h)\|, \end{aligned}$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. Now, using the condition (2.14.1) with $y = 0$, we obtain

$$D^+m(t) \leqslant g(t, m(t) + \| F(t, 0)\|,$$

and this gives, by Theorem 1.4.1, the estimate

$$\| x(t)\| \leqslant r(t), \qquad t \geqslant t_0, \tag{2.14.4}$$

where $r(t)$ is the maximal solution of (2.14.2). The conclusion follows from the hypothesis and the relation (2.14.4).

THEOREM 2.14.2. Assume that $F \in C[J \times R^n, R^n]$ and $(\partial F/\partial x)(t, x)$ exists and is continuous on $J \times R^n$. Let

$$x \cdot [H(t, x) + H^*(t, x)]x \leqslant 2\| x \| g(t, \| x \|), \tag{2.14.5}$$

where H^* is the transpose of H, which is given by

$$H(t, x) = \int_0^1 (\partial F/\partial x)(t, xs)\, ds,$$

and $g \in C[J \times R_+, R]$. Then every solution $x(t)$ of (2.14.3) tends to zero as $t \to \infty$, if every solution $u(t)$ of (2.14.2) tends to zero as $t \to \infty$. If, in particular, $F(t, 0) \equiv 0$, then, the trivial solution of (2.14.3) is asymptotically stable whenever the null solution of (2.14.2) is asymptotically stable.

Proof. If $x(t)$ is any solution of (2.14.3), write

$$m^2(t) = \| x(t)\|^2.$$

Then

$$2m(t)m'(t) = 2x(t) \cdot F(t, x(t)).$$

Since

$$F(t, x) - F(t, 0) = \int_0^1 \frac{\partial F(t, xs)}{\partial x}\, ds \cdot x,$$

using (2.14.5), we get the inequality

$$2m(t)m'(t) \leqslant 2\| x(t)\| \| F(t, 0)\| + 2\| x(t)\| g(t, \| x(t)\|),$$

which implies, arguing as in Theorem 2.13.2, that

$$\| x(t)\| \leqslant r(t), \qquad t \geqslant t_0,$$

$r(t)$ being the maximal solution of (2.14.2) such that $\| x_0 \| \leqslant u_0$. The stated result is clear from this estimate.

THEOREM 2.14.3. Let $U(t)$ be the fundamental matrix solution of

$$x' = A(t)x, \tag{2.14.6}$$

$A(t)$ being a continuous $n \times n$ matrix and $U(t_0) =$ unit matrix. Let $F \in C[J \times R^n, R^n]$, $F(t, 0) = 0$, $t \in J$, and

$$\| U^{-1}(t)F(t, U(t)y)\| \leqslant g(t, \| y \|), \tag{2.14.7}$$

where $g \in C[J \times R_+, R_+]$. Assume that the solutions $u(t) = u(t, t_0, u_0)$ of

$$u' = g(t, u), \qquad u(t_0) = u_0 \tag{2.14.8}$$

are bounded for $t \geqslant t_0$. Then, the stability properties of the linear differential system (2.14.6) imply the corresponding stability properties of the null solution of

$$x' = A(t)x + F(t, x), \qquad x(t_0) = x_0 . \tag{2.14.9}$$

Proof. The linear transformation

$$x = U(t)y$$

reduces (2.14.9) into

$$y' = U^{-1}(t)F(t, U(t)y). \tag{2.14.10}$$

Let $y(t)$ be any solution of (2.14.10) such that $y(t_0) = x_0$ and $\| x_0 \| \leqslant u_0$. Then, if $m(t) = \| y(t) \|$, it is easy to obtain, in view of (2.14.7), the differential inequality

$$D^+m(t) \leqslant g(t, m(t)),$$

and hence, by Theorem 1.4.1, we arrive at the inequality

$$\| y(t)\| \leqslant r(t), \qquad t \geqslant t_0 , \tag{2.14.11}$$

$r(t)$ being the maximal solution of (2.14.8). If $x(t)$ is any solution of (2.14.9), we deduce, from the relation (2.14.11) and the transformation $x = U(t)y$, that

$$\| x(t)\| \leqslant r(t)\| U(t)\|, \qquad t \geqslant t_0 . \tag{2.14.12}$$

Since all the solutions of (2.14.8) are assumed to be bounded, it follows from (2.14.12) that the stability properties of the null solution of (2.14.9) are implied by the corresponding stability properties of (2.14.6).

THEOREM 2.14.4. Assume that the fundamental matrix solution $U(t)$ of the linear system (2.14.6) verifies

$$\| U(t)\| \leqslant M \qquad \| U^{-1}(s)\| \leqslant M, \qquad t_0 \leqslant s \leqslant t. \tag{2.14.13}$$

Let $F \in C[J \times R^n, R^n]$ and

$$\| F(t, x)\| \leqslant \lambda(t)\| x \|, \tag{2.14.14}$$

where $\lambda \in C[J, R_+]$ and

$$\int_{t_0}^{\infty} \lambda(s)\, ds < \infty. \tag{2.14.15}$$

Then, the stability properties of the null solution of (2.14.9) depend on the corresponding stability properties of the linear system (2.14.6).

Proof. By Theorem 2.6.2, any solution $x(t)$ of (2.14.9) satisfies the integral equation

$$x(t) = U(t)x_0 + \int_{t_0}^{t} U(t)U^{-1}(s)F(s, x(s))\, ds. \tag{2.14.16}$$

The estimate (2.14.14) reduces (2.14.16) to the inequality

$$\| x(t)\| \leqslant \| U(t)\| [\| x_0 \| + \int_{t_0}^{t} \| U^{-1}(s)\| \lambda(s) \| x(s)\|\, ds],$$

which, by writing $m(t) = \| x(t)\| / \| U(t) \|$ and using (2.14.13), shows that

$$m(t) \leqslant \| x_0 \| + \int_{t_0}^{t} M^2 \lambda(s) m(s)\, ds, \qquad t \geqslant t_0 .$$

By Theorem 1.9.1, we get

$$m(t) \leqslant \| x_0 \| \exp \left[\int_{t_0}^{t} M^2 \lambda(s)\, ds \right], \qquad t \geqslant t_0 ,$$

which, because of the definition of $m(t)$ and (2.14.15), yields the estimate

$$\| x(t)\| \leqslant K \| x_0 \| \| U(t)\|, \qquad t \geqslant t_0 ,$$

for some constant $K > 0$. The conclusion is now immediate.

THEOREM 2.14.5. Let the assumptions of Theorem 2.14.4 hold except that the inequalities (2.14.13) are replaced by

$$\| U(t)\| \leqslant M, \qquad \| U(t)U^{-1}(s)\| \leqslant N, \qquad t_0 \leqslant s \leqslant t. \tag{2.14.17}$$

Then, all solutions $x(t)$ of (2.14.9) exist for $t \geqslant t_0$ and verify the estimate

$$\| x(t)\| \leqslant K \| x_0 \|, \qquad t \geqslant t_0 , \tag{2.14.18}$$

for some $K > 0$. If, in addition, $y(t)$ is the solution of the linear system (2.14.6) with $y(t_0) = x_0$ such that $\lim_{t\to\infty} y(t) = 0$, then $\lim_{t\to\infty} x(t) = 0$.

Proof. The integral equation (2.14.16) gives, using the conditions (2.14.17) and (2.14.14),

$$\| x(t)\| \leqslant M\| x_0 \| + \int_{t_0}^{t} M\lambda(s)\| x(s)\| \, ds, \qquad t \geqslant t_0 ,$$

which, by Theorem 1.9.1, leads to

$$\begin{aligned} \| x(t)\| &\leqslant M\| x_0 \| \exp \left[\int_{t_0}^{t} \lambda(s) \, ds\right] \\ &\leqslant K\| x_0 \|, \qquad t \geqslant t_0 , \end{aligned}$$

for some constant $K > 0$, proving (2.14.18).

If $y(t)$ is the solution of (2.14.6) such that $\lim_{t\to\infty} y(t) = 0$, given $\epsilon > 0$, there exists a $T(\epsilon)$ such that $\| y(t)\| < \epsilon$ for all $t \geqslant T(\epsilon)$. Hence, for $t \geqslant T(\epsilon)$, we have, successively, using (2.14.17) and (2.14.14),

$$x(t) = y(t) + \int_{t_0}^{t} U(t)U^{-1}(s)F(s, x(s)) \, ds,$$

$$\| x(t)\| \leqslant \epsilon + \int_{t_0}^{t} M\lambda(s)\| x(s)\| \, ds.$$

Again, applying Theorem 1.9.1, we obtain

$$\| x(t)\| \leqslant \epsilon \exp \int_{t_0}^{t} M\lambda(s) \, ds \leqslant k\epsilon,$$

for some constant $k > 0$, which is independent of ϵ and T. This proves that $\lim_{t\to\infty} x(t) = 0$.

THEOREM 2.14.6. Assume that (i) A is an $n \times n$ constant matrix and the characteristic roots of A have negative real parts; (ii) $F \in C[J \times R^n, R^n]$, and, given any $\epsilon > 0$, there exist $\delta(\epsilon)$, $T(\epsilon) > 0$ such that

$$\| F(t, x)\| \leqslant \epsilon\| x \|$$

provided $\| x \| \leqslant \delta(\epsilon)$ and $t \geqslant T(\epsilon)$; (iii) $G \in C[J \times R^n, R^n]$ and there exists an $\alpha > 0$ such that, if $\| x \| < \alpha$ and $t \in J$,

$$\| G(t, x)\| \leqslant \gamma(t),$$

where $\gamma \in C[J, R_+]$ and

$$p(t) = \int_{t}^{t+1} \gamma(s) \, ds \to 0 \qquad \text{as} \quad t \to \infty.$$

Then, there exist $T_0 \geqslant 0$, $\delta > 0$ such that, for every $t_0 \geqslant T_0$ and $\| x_0 \| < \delta$, any solution $x(t) = x(t, t_0, x_0)$ of the differential system

$$x' = Ax + F(t, x) + G(t, x), \qquad x(t_0) = x_0 \tag{2.14.19}$$

satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

If, in particular, (2.14.19) possesses trivial solution, then the trivial solution is asymptotically stable.

Proof. By assumption (i), it follows that there exist constants $K \geqslant 1$ and $\sigma > 0$ such that

$$\| e^{At} \| \leqslant Ke^{-\sigma t}, \qquad t \geqslant 0. \tag{2.14.20}$$

Choose ϵ so that $0 < \epsilon < \min(\sigma/K, \alpha)$. Because of assumption (ii), we can choose $T(\epsilon) \geqslant 1$ and $\delta(\epsilon) \leqslant \epsilon$. Let $T_0 \geqslant T(\epsilon)$ be so large that $t \geqslant T_0$ implies that

$$\int_1^t \exp[-(\sigma - K\epsilon)(t - s)]\gamma(s)\, ds < \delta(\epsilon)/2K = \delta_1 . \tag{2.14.21}$$

We shall prove below that such a choice is possible. Observe that

$$\begin{aligned}\int_{t_0-1}^t p(s)\, ds &= \int_{t_0-1}^t \left[\int_s^{s+1} \gamma(u)\, du\right] ds\\ &\geqslant \int_{t_0}^t \left[\int_{u-1}^u \gamma(u)\, ds\right] du\\ &= \int_{t_0}^t \gamma(u)\, du \qquad \text{for} \quad t \geqslant t_0 \geqslant 1.\end{aligned}$$

Also, for $\beta > 0$,

$$\begin{aligned}\int_{t_0}^t e^{\beta s}\gamma(s)\, ds &\leqslant \int_{t_0-1}^t \left[\int_s^{s+1} e^{\beta u}\gamma(u)\, du\right] ds\\ &\geqslant \int_{t_0-1}^t e^{\beta(s+1)} p(s)\, ds,\end{aligned}$$

whence

$$e^{-\beta t}\int_1^t e^{\beta s}\gamma(s)\, ds \leqslant e^{-\beta t}\int_0^t e^{\beta(s+1)} p(s)\, ds.$$

Applying L'Hospital's rule on

$$\frac{1}{e^{\beta t}}\left[\int_0^t e^{\beta(s+1)} p(s)\, ds\right],$$

it can be easily verified that

$$\lim_{t\to\infty} e^{-\beta t} \int_1^t e^{\beta s}\gamma(s)\, ds = 0$$

for all $\beta > 0$. The validity of (2.14.21) is now clear.

Let $t_0 \geqslant T_0$ and $\| x_0 \| < \delta_1$. Then, as long as $\| x(t)\| < \delta(\epsilon)$, we have

$$x(t) = \exp[A(t - t_0)]\, x_0 + \int_{t_0}^t e^{A(t-s)}[F(s, x(s) + G(s, x(s))]\, ds,$$

from which, using assumptions (ii), (iii), and the estimate (2.14.20), it follows that

$$\| x(t)\| e^{\sigma t} \leqslant K\delta_1 \exp(\sigma t_0) + \int_{t_0}^t [\epsilon \| x(s)\| + \gamma(s)]Ke^{\sigma s}\, ds.$$

By Corollary 1.9.1, we obtain

$$\| x(t)\| e^{\sigma t} \leqslant K\delta_1 \exp(\sigma t_0) \exp[K\epsilon(t - t_0)] + \int_{t_0}^t Ke^{\sigma s}\gamma(s) \exp[K\epsilon(t - s)]\, ds,$$

so that

$$\| x(t)\| \leqslant K\delta_1 \exp[-(\sigma - K\epsilon)(t - t_0)] + K \int_{t_0}^t \exp[-(\sigma - K\epsilon)(t - s)]\, \gamma(s)\, ds, \tag{2.14.22}$$

which, using (2.14.21), yields

$$\begin{aligned} \| x(t)\| &\leqslant K\delta_1 + K \int_1^t \exp[-(\sigma - K\epsilon)(t - s)]\, \gamma(s)\, ds \\ &\leqslant K\delta_1 + \tfrac{1}{2}\delta(\epsilon) = \delta(\epsilon). \end{aligned}$$

Thus, the inequality

$$\| x(t)\| < \delta(\epsilon) \tag{2.14.23}$$

holds on $[t_0, \infty)$, which implies that (2.14.22) is true for $t \geqslant t_0$. Hence,

$$\lim_{t\to\infty} x(t) = 0.$$

If $x = 0$ is the solution of (2.14.19), it is clearly asymptotically stable. For, we have immediately from (2.14.23) that $\| x(t)\| < \epsilon$ whenever $\| x_0 \| < \delta(\epsilon)$, since we have chosen $\delta(\epsilon) \leqslant \epsilon$. The proof is complete.

Notice that $p(t) = \int_t^{t+1} \gamma(s)\, ds \to 0$ as $t \to \infty$ holds if either $\gamma(t) \to 0$ as $t \to \infty$ or $\int_0^\infty \gamma(s)\, ds < \infty$.
This observation suggests the following:

COROLLARY 2.14.1. The conclusion of Theorem 2.14.6 is valid for the system

$$x' = Ax + F(t, x) + G_1(t, x) + G_2(t, x)$$

if conditions (i) and (ii) of Theorem 2.14.6 hold and, for small $\| x \|$, $G_1(t, x) \to 0$ as $t \to \infty$ uniformly in x and $\| G_2(t, x) \| \leqslant \lambda(t)\| x \|$, where $\lambda \in C[J, R_+]$, and $\int_0^\infty \lambda(s)\, ds < \infty$.

THEOREM 2.14.7. Assume that $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, $f_x(t, x)$ exists and is continuous on $J \times S_\rho$, and, for $t \in J$,

$$\|f_x(t, x) - f_x(t, 0)\| \leqslant K \| x \|. \tag{2.14.24}$$

Suppose that

$$\limsup_{t\to\infty} (t - t_0)^{-1} \int_{t_0}^t \mu[f_x(s, 0)]\, ds = \sigma_0 < 0. \tag{2.14.25}$$

Let $y(t)$ be the solution of the variational system

$$y' = f_x(t, x(t))y, \qquad y(t_0) = x_0, \tag{2.14.26}$$

where $x(t) = x(t, t_0, x_0)$ is the solution of

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{2.14.27}$$

existing for $t \geqslant t_0$. Then, we have the estimate

$$\| y(t)\| \leqslant \| x_0 \| \exp \left[K\eta(t - t_0) + \int_{t_0}^t \mu[f_x(s, 0)]\, ds\right] \tag{2.14.28}$$

for $t \geqslant t_0$ and $\lim_{t\to\infty} y(t) = 0$.

Proof. Consider the function $m(t) = \| y(t)\|$. Observe that

$$\frac{m(t + h) - m(t)}{h} \leqslant \frac{1}{h} [\| I + hf_x(t, x(t))\| - 1]m(t) + \frac{\| \epsilon(h)\|}{h},$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. Furthermore, from the definition of $\mu(A)$ given by (2.13.6), the inequality (2.13.10), and the assumption (2.14.24), we obtain

$$\mu[f_x(t, x(t))] \leqslant \mu[f_x(t, 0)] + K\| x(t)\|. \tag{2.14.29}$$

Hence, it follows that

$$D^+m(t) \leqslant [\mu[f_x(t, 0)] + K\| x(t)\|]m(t). \tag{2.14.30}$$

As the hypotheses of Theorem 2.13.2 are satisfied, the trivial solution of (2.14.27) is asymptotically stable, and therefore, if $\| x_0 \|$ is small enough, $\| x(t)\| < \eta$, $t \geqslant t_0$. Consequently, choosing $\| x_0 \|$ sufficiently small, we have, from (2.14.30),

$$D^+m(t) \leqslant [\mu[f_x(t, 0)] + K\eta]m(t),$$

which, by Theorem 1.4.1, leads to the estimate (2.14.28). Moreover, by condition (2.14.25), it results that, if η is small,

$$\lim_{t\to\infty} \exp \left[K\eta(t - t_0) + \int_{t_0}^{t} \mu[f_x(s, 0)]\, ds\right] = 0.$$

This, together with (2.14.28), implies that $\lim_{t\to\infty} y(t) = 0$.

REMARK 2.14.1. The condition (2.14.25) implies that the solutions $y_0(t)$ of the variational system

$$y_0' = f_x(t, 0)y_0\,, \qquad y_0(t_0) = x_0 \tag{2.14.31}$$

have the property that $\lim_{t\to\infty} y_0(t) = 0$. For, setting $m(t) = \| y_0(t)\|$, we obtain

$$D^+m(t) \leqslant \mu[f_x(t, 0)]m(t),$$

and hence, by Theorem 1.4.1,

$$\| y_0(t)\| \leqslant \| x_0 \| \exp \left[\int_{t_0}^{t} \mu[f_x(s, 0)]\, ds\right], \qquad t \geqslant t_0\,.$$

The assumption (2.14.25) assures that $\lim_{t\to\infty} y_0(t) = 0$.

Thus, in essence, Theorem 2.14.7 guarantees the asymptotic behavior of the solutions of (2.14.26), whenever there exists a similar behavior for the solutions of (2.14.31). From these considerations, we infer the following lemma, which is interesting in itself.

LEMMA 2.14.1. Let $A(t)$ be a continuous $n \times n$ matrix on J. If $x(t)$ is the solution of

$$x' = A(t)x, \qquad x(t_0) = x_0\,,$$

we have

$$\| x(t)\| \leqslant \| x_0 \| \exp \left[\int_{t_0}^{t} \mu[A(s)]\, ds\right], \qquad t \geqslant t_0\,.$$

COROLLARY 2.14.2. Under the assumptions of Theorem 2.14.7, if

$$\lim_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds = \sigma, \tag{2.14.32}$$

then $\sigma < 0$.

Proof. Since the trivial solution of (2.14.27) is asymptotically stable by Theorem 2.13.2, choosing $\| x_0 \|$ small, we can make $\| x(t)\| < \eta$, $t \geqslant t_0$, and hence we have

$$\sigma \leqslant \sigma_0 + K\eta$$

because of (2.14.29). It therefore follows that, by choosing η sufficiently small, σ can be made less than zero whenever $\sigma_0 < 0$.

THEOREM 2.14.8. Assume that $f \in C[J \times R^n, R^n]$, and $f_x(t, x)$ exists and is continuous on $J \times R^n$. Suppose further that $x(t) = x(t, t_0, x_0)$ is the solution of (2.14.27) with the property

$$\lim_{t\to\infty} x(t) = 0. \tag{2.14.33}$$

If σ defined by the relation (2.14.32) is less than zero, then every solution $x(t, t_0, y_0)$ of (2.14.27) such that x_0, y_0 belong to a convex subset of R^n satisfies

$$\lim_{t\to\infty} x(t, t_0, y_0) = 0. \tag{2.14.34}$$

Proof. Let $x(t, t_0, x_0)$, $x(t, t_0, y_0)$ be the solutions of (2.14.27) such that x_0, y_0 belong to a convex subset of R^n. Then, by Theorem 2.6.4, we get

$$x(t, t_0, y_0) = x(t, t_0, x_0) + \left[\int_0^1 \Phi(t, t_0, x_0 + s(y_0 - x_0))\, ds\right] [y_0 - x_0],$$

and hence, for $t \geqslant t_0$,

$$\| x(t, t_0, y_0)\| \leqslant \| x(t, t_0, x_0)\| + \| y_0 - x_0 \| \exp\left[\int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds\right],$$

by virtue of Lemma 2.14.1. The relation (2.14.32) implies that

$$\int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds \leqslant \frac{\sigma}{2}(t - t_0)$$

for sufficiently large t, and therefore the assumption $\sigma < 0$ yields

$$\lim_{t\to\infty} \exp\left[\int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds\right] = 0.$$

This, together with (2.14.33), assures (2.14.34).

Corollary 2.14.3. In addition to the hypotheses of Theorem 2.14.8, if we assume that $f(t, 0) \equiv 0$, then the trivial solution of (2.14.27) is asymptotically stable.

Theorem 2.14.9. Let $f \in C[J \times R^n, R^n]$, $f(t, 0) \equiv 0$, and $f_x(t, x)$ exist and be continuous on $J \times R^n$. Let $G \in C[J \times R^n, R^n]$, and, given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\| G(t, y)\| \leqslant \epsilon \| y \|,$$

provided $\| y \| \leqslant \delta(\epsilon)$. Assume that the trivial solution of (2.14.27) is asymptotically stable and that σ defined by (2.14.32) is negative. Then, the trivial solution of the perturbed system

$$y' = f(t, y) + G(t, y), \qquad y(t_0) = y_0 \tag{2.14.35}$$

is asymptotically stable.

Proof. Let $x(t) = x(t, t_0, y_0)$, $y(t) = y(t, t_0, y_0)$ be the solutions of (2.14.27) and (2.14.35), respectively. Then, by Theorem 2.6.3,

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t, s, y(s)) G(s, y(s))\, ds, \qquad t \geqslant t_0 .$$

Moreover, by Theorem 2.6.4, we infer that

$$x(t) = \left[\int_0^1 \Phi(t, t_0, sy_0)\, ds\right] y_0 .$$

The preceding two relations, in view of Lemma 2.14.1, lead to the inequality

$$\| y(t)\| \leqslant \| y_0 \| \exp\left[\int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds\right]$$
$$+ \int_{t_0}^{t} \exp\left[\int_s^t \mu[f_x(s_1, x(s_1))]\, ds_1\right] \| G(s, y(s))\|\, ds$$

Let $\epsilon > 0$ be given. By assumption on $G(t, y)$, it follows that

$$\| y(t)\| \leqslant \| y_0 \| \exp\left[\int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds\right]$$
$$+ \epsilon \int_{t_0}^{t} \exp\left[\int_s^t \mu[f_x(s_1, x(s_1))]\, ds_1\right] \| y \|\, ds,$$

provided $\| y(t)\| \leqslant \delta(\epsilon)$. Theorem 1.9.1 readily gives the estimate

$$\| y(t)\| \leqslant \| y_0 \| \exp \left[\epsilon(t - t_0) + \int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds\right], \qquad (2.14.36)$$

for $t \geqslant t_0$, which implies that $\| y(t)\|$ remains less than $\delta(\epsilon)$ if $\| y_0 \|$ is small enough, because the condition $\sigma < 0$ shows that, for sufficiently small $\epsilon > 0$,

$$\lim_{t\to\infty} \exp \left[\epsilon(t - t_0) + \int_{t_0}^{t} \mu[f_x(s, x(s))]\, ds\right] = 0.$$

Thus, (2.14.36) is valid for $t \geqslant t_0$, and the asymptotic stability of the trivial solution of (2.14.35) follows.

THEOREM 2.14.10. Assume that (i) $f \in C[J \times R^n, R^n]$, $f(t, 0) \equiv 0$, and $f_x(t, x)$ is continuous on $J \times R^n$; (ii) $\mu[f_x(t, 0)] \leqslant -\sigma$, $\sigma > 0$, $t \in J$; (iii) $G \in C[J \times R^n, R^n]$, $G(t, 0) \equiv 0$, and there exists an $\alpha > 0$ such that, if $\| x \| < \alpha$, $t \in J$, $\| G(t, x)\| \leqslant \gamma(t)$, where $\gamma \in C[J, R_+]$ and

$$p(t) = \int_{t}^{t+1} \gamma(s)\, ds \to 0 \qquad \text{as} \quad t \to \infty.$$

Then, the trivial solution of (2.14.35) is asymptotically stable.

Proof. Let $\epsilon > 0$ be given such that $0 < \epsilon < \min(\sigma, \alpha)$. Choose $T_0 \geqslant 1$ so large that, for $t \geqslant T_0$, we have

$$\int_{1}^{t} \exp[-(\sigma - \epsilon)(t - s)]\gamma(s)\, ds < \frac{\delta(\epsilon)}{2} = \delta_1, \qquad (2.14.37)$$

where $\delta(\epsilon) \leqslant \epsilon$. This choice is possible, as shown in Theorem 2.14.6. It is easy to show that, whenever $\| x_0 \| < \delta_1$, $\| x(t)\| < \delta(\epsilon)$, $t \geqslant t_0$. For, otherwise, there would exist a $t_1 > t_0 \geqslant T_0$ such that

$$\| x(t_1)\| = \delta(\epsilon), \qquad \| x(t)\| \leqslant \delta(\epsilon), \qquad t \in [t_0, t_1].$$

Define $m(t) = \| x(t) \|$. Then, for $t \in [t_0, t_1]$, we obtain the differential inequality

$$\begin{aligned} m'_+(t) &\leqslant \mu[f_x(t, 0)]m(t) + \| F(t, x(t))\| + \| G(t, x(t))\| \\ &\leqslant -(\sigma - \epsilon)m(t) + \gamma(t). \end{aligned}$$

Here we have used assumptions (ii) and (iii) of the theorem, in addition to the relations (2.13.12) and (2.13.13) and the argument employed in Theorem 2.13.2.

An application of Theorem 1.4.1 gives

$$\| x(t)\| \leqslant \| x_0 \| \exp[-(\sigma - \epsilon)(t - t_0)] + \int_{t_0}^{t} \exp[-(\sigma - \epsilon)(t - s)]\, \gamma(s)\, ds \tag{2.14.38}$$

for $t \in [t_0, t_1]$. At $t = t_1$, there arises an absurdity

$$\begin{aligned} \delta(\epsilon) &< \delta_1 + \int_1^{t_1} \exp[-(\sigma - \epsilon)(t_1 - s)]\, \gamma(s)\, ds \\ &< \delta_1 + \delta_1 = \delta(\epsilon), \end{aligned}$$

because of (2.14.37). This proves that, if $\| x_0 \| < \delta_1$, $\| x(t)\| < \delta(\epsilon)$, $t \geqslant t_0$, which, in its turn, implies the inequality (2.14.38) for all $t \geqslant t_0$. Since $\delta(\epsilon) \leqslant \epsilon$, the stated result follows, as in Theorem 2.14.6.

COROLLARY 2.14.4. The function $f(t, x) = Ax + F(t, x)$, where A is an $n \times n$ constant matrix such that $\mu(A) \leqslant -\sigma$ and $F(t, x)$ satisfies assumption (ii) of Theorem 2.14.6, is admissible in Theorem 2.14.10.

2.15. Periodic and almost periodic systems

We shall be concerned in this section with the existence of periodic and almost periodic solutions of differential equations. Let us first state the following:

LEMMA 2.15.1. Let E be an inductively ordered set, and let T be a transformation from E into E such that, for any $x \in E$, we have $T(x) \geqslant x$. Then, there exists at least one point $x \in E$ satisfying $T(x) = x$.

As an application of this lemma, we prove an existence theorem for periodic solutions.

THEOREM 2.15.1. Assume that (i) $g \in C[J \times R_+, R]$, $g(t, u)$ is nondecreasing in u for each $t \in J$, periodic in t with a period ω, and the differential equation

$$u' = g(t, u) \tag{2.15.1}$$

admits a periodic solution of period ω; (ii) $f \in C[J \times R^n, R^n]$, $f(t, x)$ is periodic in t with a period ω, and, for $t \in J$, $x, y \in R^n$, and sufficiently small $h > 0$,

$$\| x - y + h[f(t, x) - f_1(t, y)]\| \leqslant \| x - y \| + hg(t, \| x - y \|) + 0(h), \tag{2.15.2}$$

where $f_1 \in C[J \times R^n, R^n]$; (iii) the functions f, f_1, and g are smooth enough to assure the existence and uniqueness of solutions, and the system

$$y' = f_1(t, y) \tag{2.15.3}$$

has a bounded nondecreasing solution. Then, the differential system

$$x' = f(t, x) \tag{2.15.4}$$

admits a periodic solution of period ω.

Proof. Let $y(t)$ be the bounded monotonic solution of (2.15.3) such that $y(t_0) = y_0$, $t_0 \geqslant 0$. Suppose that $u(t)$ is the periodic solution of (2.15.1) of period ω. It is possible to choose t_0 and u_0 such that

$$u(t_0) = u_0 \geqslant 0, \qquad u(t) - u_0 \geqslant 0, \qquad t \geqslant t_0. \tag{2.15.5}$$

Define $m(t) = \| x(t) - y(t)\|$, where $x(t)$ is the solution of (2.15.4) with the property $x(t_0) = y_0$. Clearly, $m(t_0) = 0$. We have, using the condition (2.15.2), the differential inequality

$$D^+m(t) \leqslant g(t, m(t)).$$

Consider the solutions $u(t, \epsilon)$ of

$$u' = g(t, u) + \epsilon, \qquad u(t_0, \epsilon) = u_0 + \epsilon$$

for sufficiently small $\epsilon > 0$. Setting

$$p(t, \epsilon) = u(t, \epsilon) - (u_0 + \epsilon),$$

because of the nondecreasing nature of $g(t, u)$ we get

$$p'(t, \epsilon) > g(t, p(t, \epsilon)).$$

According to Theorem 1.2.1, we infer that

$$m(t) < p(t, \epsilon), \qquad t \geqslant t_0.$$

But

$$\lim_{\epsilon \to 0} p(t, \epsilon) = \lim_{\epsilon \to 0} [u(t, \epsilon) - (u_0 + \epsilon)] = u(t) - u_0$$

uniformly in t, and therefore

$$\| x(t) - y(t)\| \leqslant u\ t) - u_0, \qquad t \geqslant t_0. \tag{2.15.6}$$

Define the point $x(t_0)$by p_0. Take the point $x(t_0 + \omega)$ on the solution $x(t)$ and denote it by p_1. Let T be the transformation that takes any point p_0 to p_1 by the foregoing process. Since the function $f(t, x)$ is periodic in t with a period ω, any solution passing through the fixed point, under the preceding transformation, is clearly a periodic solution. Hence, it is enough to show the existence of a fixed point, under the transformation T defined previously. Since $u(t)$ is periodic and $u(t_0) = u_0$, we have

$$u(t_0 + n\omega) - u_0 = 0, \qquad n = 0, 1, 2,\ldots.$$

It therefore follows from (2.15.6) that

$$x(t_0 + n\omega) = y(t_0 + n\omega), \qquad n = 0, 1, 2,\ldots.$$

By assumption, $y(t)$ is a bounded monotonic solution of (2.15.3). Thus, it is evident that the points $x(t_0 + n\omega)$ form a bounded, monotonic, and denumerably infinite set. If their upper bound is also included, the set becomes inductively ordered. Hence, the application of Lemma 2.15.1 yields a fixed point, and the theorem is proved.

COROLLARY 2.15.1. If, in addition to the hypotheses of Theorem 2.15.1, $f_1(t, y)$ is periodic in t with a period ω, the assertion of Theorem 2.15.1 remains valid. In particular, $f_1(t, y) \equiv 0$ is admissible.

We remark that the monotony of $g(t, u)$ can be dispensed with if the periodic solution $u(t)$ of (2.15.1) has the property that, for some $t_0 \geqslant 0$, $u(t_0) = 0$ and $u(t) \geqslant 0$ for $t \geqslant 0$.

Another set of sufficient conditions for the existence of a periodic solution is given by

THEOREM 2.15.2. Assume that (i) $g \in C[J \times R_+, R]$, and the differential equation

$$u' = g(t, u), \qquad u(0) = u_0 \geqslant 0 \tag{2.15.7}$$

has unique solution $u(t, 0, u_0)$ such that

$$\lim_{t\to\infty} u(t, 0, u_0) = 0;$$

(ii) $f \in C[J \times R^n, R^n]$, and $f(t, x)$ is periodic in t with a period ω and is smooth enough to guarantee the existence and uniqueness of solutions of the system (2.15.4), and, for $t \in J$, $x, y \in R^n$,

$$\| x - y + h[f(t, x) - f(t, y)]\| \leqslant \| x - y \| + hg(t, \| x - y \|) + 0(h) \tag{2.15.8}$$

for sufficiently small $h > 0$; (iii) the system (2.15.4) has a bounded solution existing on $[0, \infty)$. Then, the system (2.15.4) admits a periodic solution of period ω.

Proof. Let $x_0(t) = x_0(t, 0, x_0)$ be the bounded solution of (2.15.4) defined for $t \geqslant 0$. We shall show that, under the assumptions of the theorem, we have

$$\lim_{t\to\infty} x_0(t + \omega) - x_0(t) = 0. \tag{2.15.9}$$

Let $z(t) = x_0(t + \omega) - x_0(t)$. Then,

$$\begin{aligned} z'(t) &= f(t + \omega, x_0(t + \omega)) - f(t, x_0(t)) \\ &= f(t, x_0(t) + z(t)) - f(t, x_0(t)), \end{aligned}$$

because of the periodicity of $f(t, x)$ in t. Hence, setting $m(t) = \| z(t)\|$, we observe, for small $h > 0$, that

$$\begin{aligned} m(t + h) \leqslant \| x_0(t + \omega) - x_0(t) + h[f(t, x_0(t) + z(t)) \\ - f(t, x_0(t))]\| + \| \epsilon(h)\|, \end{aligned}$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. Using condition (2.15.8), it is easy to get the inequality

$$D^+m(t) \leqslant g(t, m(t)),$$

which implies, by Theorem 1.4.1, choosing $u_0 = \| x_0(\omega) - x_0 \|$,

$$\| x_0(t + \omega) - x_0(t)\| \leqslant u(t, 0, \| x_0(\omega) - x_0 \|), \qquad t \geqslant 0,$$

$u(t, 0, u_0)$ being the unique solution of (2.15.7). Consequently, in view of assumption (i), the relation (2.15.9) is valid.

Since $x_0(t)$ is bounded, the sequence $\{x_n\} = \{x_0(n\omega)\}$ is bounded. Hence, a subsequence $\{x_{n_k}\}$ can be extracted which converges to a point x^*. It follows from (2.15.9) that, for any n,

$$\lim_{n\to\infty} x_0[(n + 1)\omega] - x_0(n\omega) = 0,$$

and thus,

$$\lim_{k\to\infty} x_{n_k+1} = x^*. \tag{2.15.10}$$

Observing that the functions defined by $x(t, 0, x_n) = x_0(t + n\omega)$ are the solutions of (2.15.4) through $(0, x_n)$, where $x_n = x_0(n\omega)$, in view of the periodicity of $f(t, x)$ in t, we see that the fact

$$\lim_{k\to\infty} x_{n_k} = x^*$$

implies

$$\lim_{k\to\infty} x(t, 0, x_{n_k}) = x(t, 0, x^*),$$

and hence, from (2.15.10), we obtain

$$\lim_{k\to\infty} x(t, 0, x_{n_k+1}) = x(t, 0 \ x^*).$$

However, because of the uniqueness of solutions,

$$\begin{aligned} x(t, 0, x_{n_k+1}) &= x(t, 0, x_0[(n_k + 1)\omega]) \\ &= x_0(t + (n_k + 1)\omega) \\ &= x_0(t + \omega + n_k\omega) \\ &= x(t + \omega, 0, x_0(n_k\omega)). \end{aligned}$$

Since

$$\begin{aligned} \lim_{k\to\infty} x(t, 0 \ x_{n_k+1}) &= \lim_{k\to\infty} x(t + \omega, 0, x_{n_k}) \\ &= x(t + \omega, 0, x^*), \end{aligned}$$

we have

$$x(t, 0, x^*) = x(t + \omega, 0, x^*).$$

This means that the solution that satisfies the initial condition $x(0) = x^*$ is periodic with period ω. The proof is complete.

An analogous result holds for almost periodic systems. We shall start with the following

DEFINITION 2.15.1. A function $f \in C[(-\infty, \infty) \times R^n, R^n]$ is said to be *almost periodic* in t uniformly with respect to $x \in S$, for any compact subset $S \subset R^n$, if, given any $\eta > 0$, it is possible to find a $1(\eta)$ such that, in any interval of length $1(\eta)$, there is a τ such that the inequality

$$\| f(t + \tau, x) - f(t, x) \| \leqslant \eta$$

is satisfied for $t \in (-\infty, \infty)$, $x \in S$.

THEOREM 2.15.3. Let $f \in C[(-\infty, \infty) \times R^n, R^n]$, and $f(t, x)$ be almost periodic in t uniformly with respect to $x \in S$, for any compact set $S \subset R^n$, and be smooth enough to ensure the existence and uniqueness of solutions of (2.15.4). Furthermore, suppose that, for sufficiently small $h > 0$,

$$\| x - y + h[f(t, x) - f(t, y)] \| \leqslant \| x - y \| [1 - \alpha h] + 0(h), \qquad t \geqslant 0, \tag{2.15.11}$$

where $\alpha > 0$, and that the almost periodic system (2.15.4) admits a bounded solution $x(t, t_0, x_0)$ with a uniform bound B. Then, (2.15.4) admits an almost periodic solution that is uniformly asymptotically stable.

Proof. Let $x(t)$ be the bounded solution of (2.15.4), defined on $[t_0, \infty)$ so that $\| x(t)\| \leqslant B$, $t \geqslant t_0$, where B does not depend on t_0. Let τ_k be a sequence of numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$$f(t + \tau_k, x) - f(t, x) \to 0$$

uniformly for $t \in (-\infty, \infty)$ and $x \in S$, any compact set in R^n.

For any β, let $k_0 = k_0(\beta)$ be the smallest value of k for which $\tau_{k_0} + \beta \geqslant t_0$. We have

$$\| x(t + \tau_k)\| \leqslant B, \qquad t \geqslant \beta, \qquad k \geqslant k_0(\beta).$$

We shall now show that the sequence of functions $\{x(t + \tau_k)\}$, $k \geqslant k_0(\beta)$ converges to a continuous bounded function $w(t)$ defined on $[\beta, \infty)$, with a bound independent of β, and that the convergence is uniform on all compact subsets of $[\beta, \infty)$. Since the boundedness of $x(t)$ is uniform with respect to t_0, it is sufficient to prove that the sequence $\{x(t + \tau_k)\}$, $k \geqslant k_0(\beta)$ forms a Cauchy sequence on any compact subset of $[\beta, \infty)$.

Let U be any compact subset of $[\beta, \infty)$, and let $\epsilon > 0$ be given. Choose an integer $n_0 = n_0(\epsilon, \beta) \geqslant k_0$ so large that, for $k_1 \geqslant n_0$,

$$\begin{aligned} &\text{(i)} \quad \exp[-\alpha(\beta + \tau_{k_1} - t_0)] < \epsilon/4B, \\ &\text{(ii)} \quad \| f(t + \tau_{k_1}, x) - f(t, x)\| \leqslant \epsilon\alpha/8, \end{aligned} \tag{2.15.12}$$

for all $t \in (-\infty, \infty)$, $x \in S$. Consider the function

$$m(t) = \| x(t + \tau_{k_1}) - x(t + \tau_{k_2}) \|, \qquad k_2 \geqslant k_1 \geqslant n_0, \qquad t \geqslant t_0.$$

Setting $t_1 = t + \tau_{k_1}$, $t_2 = t + \tau_{k_2}$, we have

$$\begin{aligned} D^+m(t) \leqslant \limsup_{h\to 0^+} h^{-1}[\| x(t_1 + h) - x(t_2 + h)\| - \| x(t_1) - x(t_2) \\ + h\{f(t + \theta, x(t_1)) - f(t + \theta, x(t_2))\}\|] \\ + \limsup_{h\to 0^+} h^{-1}[\| x(t_1) - x(t_2) + h\{f(t + \theta, x(t_1)) \\ - f(t + \theta, x(t_2))\}\| - \| x(t_1) - x(t_2)\|], \end{aligned}$$

where θ is an $\epsilon\alpha/8$-translation number of $f(t, x)$ for $x \in S$ such that $t_0 + \theta \geqslant 0$, that is,

$$\| f(t + \theta, x) - f(t, x)\| \leqslant \epsilon\alpha/8, \tag{2.15.13}$$

where $x \in S$, any compact set in R^n, and $t \in (-\infty, \infty)$. In view of the condition (2.5.11), we deduce

$$\begin{aligned} D^+m(t) &\leqslant \|f(t_1, x(t_1)) - f(t+\theta, x(t_1))\| + \|f(t_2, x(t_2)) \\ &\quad - f(t+\theta, x(t_2))\| - \alpha m(t) \\ &\leqslant \|f(t_1, x(t_1)) - f(t_1+\theta, x(t_1))\| + \|f(t_1+\theta, x(t_1)) \\ &\quad - f(t+\theta, x(t_1))\| + \|f(t_2, x(t_2)) - f(t_2+\theta, x(t_2))\| \\ &\quad + \|f(t_2+\theta, x(t_2)) - f(t+\theta, x(t_2))\| - \alpha m(t), \end{aligned}$$

which, because of relations (2.15.12) and (2.15.13), yields

$$D^+m(t) \leqslant -\alpha m(t) + (\epsilon\alpha/2), \qquad t \geqslant t_0 .$$

By Theorem 1.4.1, we get

$$m(t) \leqslant m(t_0 - \tau_{k_1}) \exp[-\alpha(t + \tau_{k_1} - t_0)] + (\epsilon/2)[1 - \exp\{-\alpha(t + \tau_{k_1} - t_0)\}].$$

This, along with the boundedness of $x(t)$, for $t \geqslant t_0$, the fact that $\beta + \tau_{k_1} \geqslant t_0$, and the relation (2.15.12), implies that

$$\begin{aligned} m(t) &\leqslant 2B \exp[-\alpha(\beta + \tau_{k_1} - t_0)] + (\epsilon/2) \\ &\leqslant (\epsilon/2) + (\epsilon/2) = \epsilon, \qquad t \in U, \qquad k_2 \geqslant k_1 \geqslant n_0 . \end{aligned}$$

This proves the existence of a function $w(t)$ defined on $[\beta, \infty)$ and bounded by B. Since β is arbitrary, $w(t)$ is defined on $(-\infty, \infty)$, and we have

$$x(t + \tau_k) - w(t) \to 0 \qquad \text{as} \quad k \to \infty,$$

uniformly on all compact subsets of $(-\infty, \infty)$.

Next we show that $w(t)$ is differentiable and that it satisfies (2.15.4). Observe that $\lim_{k\to\infty} x'(t + \tau_k)$ exists uniformly on all compact subsets of $(-\infty, \infty)$, and, consequently,

$$\begin{aligned} \lim_{k\to\infty} x'(t + \tau_k) &= \lim_{k\to\infty} \lim_{h\to 0} \frac{1}{h} [x(t + \tau_k + h) - x(t + \tau_k)] \\ &= \lim_{h\to 0} \lim_{k\to\infty} \frac{1}{h} [x(t + \tau_k + h) - x(t + \tau_k)] \\ &= \lim_{h\to 0} \frac{1}{h} [w(t + h) - w(t)], \end{aligned}$$

which proves that $w'(t)$ exists. Also,

$$\begin{aligned} w'(t) &= \lim_{k\to\infty} x'(t+\tau_k) \\ &= \lim_{k\to\infty} f(t+\tau_k\,, x(t+\tau_k)) \\ &= \lim_{k\to\infty} [f(t+\tau_k\,, x(t+\tau_k)) - f(t+\tau_k\,, w(t))] \\ &\quad + \lim_{k\to\infty} [f(t+\tau_k\,, w(t))] \\ &= f(t, w(t)), \end{aligned}$$

proving that $w(t)$ is the solution of (2.15.4).

It remains to be shown that $w(t)$ is almost periodic. For this purpose, we need to show that, for any $\epsilon > 0$ and any τ for which

$$\|f(t+\tau, x) - f(t, x)\| < \epsilon$$

uniformly on $(-\infty, \infty)$ and all $x \in S$, we have

$$\| w(t+\tau) - w(t)\| < \epsilon L$$

uniformly for $t \in (-\infty, \infty)$, where L is some constant independent of ϵ and τ.

Suppose γ is such that

$$e^{-\alpha\gamma} < \epsilon/2B\alpha. \tag{2.15.14}$$

Let $z(t) = w(t+\tau) - w(t)$, so that

$$z'(t) = f(t+\tau, w(t+\tau)) - f(t, w(t)).$$

Let θ be an ϵ-translation number of $f(t, x)$ for $x \in S$, any compact set in R^n, such that $t - \gamma + \theta \geqslant 0$, that is,

$$\|f(t+\theta, x) - f(t, x)\| < \epsilon, \qquad x \in S, \qquad t \in (-\infty, \infty).$$

If $m(t) = \| z(t)\|$, we get, for small $h > 0$,

$$\begin{aligned} m(t+h) \leqslant{}& \| w(t+\tau) - w(t) + h[f(t+\theta, w(t+\tau)) - f(t+\theta, w(t))]\| \\ &+ h\|f(t+\theta, w(t+\tau)) - f(t, w(t+\tau))\| \\ &+ h\|f(t+\theta, w(t)) - f(t, w(t))\| \\ &+ h\|f(t+\tau, w(t+\tau)) - f(t, w(t+\tau))\| + \| \epsilon(h)\|, \end{aligned}$$

which, as before, yields the differential inequality

$$D^+m(t) \leqslant -\alpha m(t) + 3\epsilon,$$

whence, by Theorem 1.4.1, we have

$$\| w(t + \tau) - w(t)\| \leqslant \| w(t - \gamma + \tau) - w(t - \gamma)\| e^{-\alpha\gamma} + \frac{3\epsilon}{\alpha}[1 - e^{-\alpha\gamma}].$$

The boundedness of $w(t)$ and the relation (2.15.14) show that

$$\| w(t + \tau) - w(t)\| < \epsilon L$$

uniformly in t, where $L = 4/\alpha$.

The uniform asymptotic stability of $w(t)$ can be easily verified. This completes the proof

COROLLARY 2.15.2. Assume that (i) $A(t)$ is a continuous $n \times n$ matrix on $(-\infty, \infty)$, almost periodic in t, and

$$\mu[A(t)] \leqslant -\alpha, \qquad \alpha > 0, \qquad t \geqslant 0;$$

(ii) $f \in C[(-\infty, \infty), R^n]$, and $f(t)$ is almost periodic in t and is bounded. Then, the system

$$x' = A(t)x + f(t) \tag{2.15.15}$$

admits an almost periodic solution that is uniformly asymptotically stable.

Proof. It is easy to check that the assumptions of Theorem 2.15.3 are satisfied except the existence of a bounded solution. Hence, it only needs to be verified that (2.15.15) has a bounded solution $x(t, t_0, x_0)$ with a uniform bound B for $t \geqslant t_0$. In fact, under the assumptions, it turns out that all the solutions $x(t, t_0, x_0)$, $t_0 \in (-\infty, \infty)$, $x_0 \in R^n$ are uniformly bounded.

Let

$$\sup_{-\infty < t < \infty} \| f(t)\| = B_1,$$

and let $x(t) = x(t, t_0, x_0)$ be any solution of (2.15.15) such that $t_0 \in (-\infty, \infty)$ and $\| x_0 \| \leqslant B_1/\alpha$. Then, it can be shown that

$$\| x(t)\| < 5B_1/2\alpha = B, \qquad t \geqslant t_0.$$

If this were not true, there would exist a $t_1 > t_0$ such that

$$\| x(t_1)\| = B, \qquad \| x(t)\| \leqslant B, \qquad t \in [t_0, t_1]. \tag{2.15.16}$$

Let θ be a $B_1/3$-translation number of $A(t)x + f(t)$ for $x \in S$, a compact set in R^n, such that $t_0 + \theta \geqslant 0$, that is,

$$\|[A(t+\theta) - A(t)]x(t) + f(t+\theta) - f(t)\| < \tfrac{1}{3}B_1, \qquad (2.15.17)$$

for $t \in (-\infty, \infty)$. Defining $m(t) = \| x(t)\|$, we have, for small $h > 0$,

$$\begin{aligned} m(t+h) &\leqslant \| I + hA(t+\theta)\| \| x(t)\| + h\|f(t+\theta)\| \\ &\quad + h\|[A(t+\theta) - A(t)]x(t) + f(t+\theta) - f(t)\| + \| \epsilon(h)\|, \end{aligned}$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. Observing that $t + \theta \geqslant 0$, assumption (i), together with the relations (2.15.16) and (2.15.17), yields

$$D^+m(t) \leqslant -\alpha m(t) + \tfrac{4}{3}B_1, \qquad t \in [t_0, t_1].$$

Hence, by Theorem 1.4.1,

$$\begin{aligned} m(t) &\leqslant m(t_0)\exp[-\alpha(t - t_0)] \\ &\quad + \frac{4B_1}{3\alpha}[1 - \exp\{-\alpha(t - t_0)\}], \qquad t \in [t_0, t_1], \end{aligned}$$

which leads to an absurdity, using the fact that $\| x_0 \| \leqslant B_1/\alpha$ and (2.15.16);

$$\frac{5B_1}{2\alpha} = m(t_1) = \| x(t_1)\| \leqslant \frac{B_1}{\alpha} + \frac{4B_1}{3\alpha} = \frac{7B_1}{3\alpha}.$$

This proves the uniform boundedness of solutions of (2.15.15) and establishes the corollary.

2.16. Notes

The results of Sect. 2.1 are due to Stokes [1]. For a more general global existence theorem using Tychonoff's fixed point theorem, see Corduneanu [1].

Theorem 2.2.1 is given by O. Perron, (see Kamke (1]). The proof of Corollary 2.2.1 is new. The general uniqueness theorem 2.2.2 is due to Kamke [1]. The proof given in the text is based on that of Olech [4]. For Theorem 2.2.3, see Lakshmikantham [12] and Olech [4]. Corollary 2.2.2 is a result of Wintner [15]. Corollaries 2.2.4 and 2.2.5 are Nagumo's and Osgood's uniqueness criteria, respectively. Theorem 2.2.4 is due to Brauer [2]. Theorem 2.2.5 and the proof of Theorem 2.2.4 are taken from Walter [2]. Corollary 2.2.6 is a result of Krasnosel'skii and Krein [2].

See also Brauer [1], Kooi [1], and Luxemburg [1]. Beginning from nonuniqueness Theorem 2.2.7, the remaining results of Sect. 2.2 are due to Lakshmikantham [5, 12]. The proof of Theorem 2.3.1 is taken from Wazewski [9], whereas the second proof of Theorem 2.3.1 is from Hartman [5]. See Olech and Plis [1] for the proof that the monotony assumption in Theorem 2.3.1 cannot be dropped in general. See also Bihari [1], Cafiero [2, 4], Coddington and Levinson [1], Diaz and Walter [1], Dieudonne [1], LaSalle [1], and Viswanatham [1].

The results of Sect. 2.4 are taken from Chaplygin [1] and Lusin [1]. Lemma 2.5.1 and Theorem 2.5.1 are new: the idea is taken from Turowicz [1]. Theorem 2.5.2 is adopted from Antosiewicz [7]. The rest of the results of Sect. 2.5 are taken from Hartman [6].

Theorems 2.6.3 and 2.6.4 are due to Alekseev [1]. Most of the results of Sects. 2.7 and 2.8 are adopted from Lakshmikantham [7] and Walter [3]. See also Bihari [2], Brauer [5], and Langenhop [1].

Section 2.9 contains the work of Brauer [11]. Theorem 2.10.1 is from Lakshmikantham and Onuchic [1]. Theorems 2.10.2 and 2.10.3 are adopted from Brauer [4, 12]. See also Cesari [1] and Wintner [4, 5, 7–9]. Theorems 2.11.1 and 2.11.2 are due to Wazewski [2] and are very useful in the study of differential equations. The proofs are taken from Hartman [5]. Theorem 2.12.1 is due to Lakshmikantham and Onuchic [1]. For further results and references on the application of Wazewski's topological principle, see Cesari [1] and Hartman [5].

Theorem 2.13.1 is adopted from Lakshmikantham [7], whereas Theorem 2.13.2 is due to Brauer [10]. For Theorem 2.13.3, see Conti and Sansone [1].

Theorems 2.14.1 and 2.14.3 are adopted from Lakshmikantham [7]. For Theorem 2.14.2, see Krasovskii [4] and Zubov [1]. See Cesari [1] for Theorems 2.14.4 and 2.14.5. Theorem 2.14.6 is due to Strauss and Yorke [1]. Theorems 2.14.7, 2.14.8, and 2.14.9 are taken from Brauer [10]. See also Cesari [1], Coppel [1], Hartman [5], and Markus and Yamabe [1].

Theorem 2.15.1 is due to Lakshmikantham [7]. Theorem 2.15.2 is new. The proof uses a result from Halanay [2]. Theorem 2.15.3 is new. See also Cartwright [1], Deysach and Sell [1], Massera [2], Miller [3], Seifert [5, 6], and Sell [4].

Chapter 3

3.0. Introduction

As is well known, Lyapunov's second method has its origin in three simple theorems that form the core of what he himself called his second method for dealing with questions of stability. It is widely recognized, today, as an indispensable tool not only in the theory of stability but also in studying many other qualitative properties of solutions of differential equations. The main characteristic of this method is the introduction of a function, namely the Lyapunov function, which defines a generalized distance from the origin of the motion space. As a result, the concept of Lyapunov function, together with the theory of differential inequalities, furnishes a very general comparison principle under much less restrictive assumptions. We present, in this chapter, a variety of qualitative problems bringing out the real significance of the comparison technique.

3.1. Basic comparison theorems

Consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \in J, \tag{3.1.1}$$

where $f \in C[J \times R^n, R^n]$. For $V \in C[J \times R^n, R_+]$, we define the function

$$D^+V(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)] \tag{3.1.2}$$

for $(t, x) \in J \times R^n$. Occasionally, we write $D^+V(t, x)_{(3.1.1)}$ to denote that the definition of $D^+V(t, x)$ is with respect to the system (3.1.1).

We can now formulate the basic comparison theorems.

THEOREM 3.1.1. Let $V \in C[J \times R^n, R_+]$ and $V(t, x)$ be locally Lipschitzian in x. Assume that the function $D^+V(t, x)$ of (3.1.2) satisfies

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times R^n, \tag{3.1.3}$$

where $g \in C[J \times R_+, R]$. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of the scalar differential equation

$$u' = g(t, u), \qquad u(t_0) = u_0 \geqslant 0, \tag{3.1.4}$$

existing to the right of t_0. If $x(t) = x(t, t_0, x_0)$ is any solution of (3.1.1) existing for $t \geqslant t_0$ such that

$$V(t_0, x_0) \leqslant u_0, \tag{3.1.5}$$

then

$$V(t, x(t)) \leqslant r(t), t \geqslant t_0. \tag{3.1.6}$$

Proof. Let $x(t)$ be any solution of (3.1.1) defined for $t \geqslant t_0$ such that (3.1.5) holds. Define

$$m(t) = V(t, x(t)),$$

so that $m(t_0) \leqslant u_0$. For sufficiently small $h > 0$, we have

$$m(t + h) - m(t) = V(t + h, x(t) + hf(t, x(t)) + \epsilon(h)) - V(t, x(t)),$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. Since $V(t, x)$ is locally Lipschitzian in x, we get, using (3.1.3), the inequality

$$D^+m(t) \leqslant g(t, m(t)), \qquad t \in J.$$

Applying Theorem 1.4.1, we obtain the desired result (3.1.6).

REMARK 3.1.1. Let

$$S_\rho = [x \in R^n : \| x \| < \rho],$$

and assume that the condition (3.1.3) holds for $(t, x) \in J \times S_\rho$. If $x(t)$ is any solution of (3.1.1) such that $\| x_0 \| < \rho$, then (3.1.5) implies (3.1.6) as far as $x(t)$ remains in S_ρ to the right of t_0.

COROLLARY 3.1.1. If the function $g(t, u)$ in Theorem 3.1.1 is identically zero, i.e.,

$$D^+V(t, x) \leqslant 0, \qquad (t, x) \in J \times R^n,$$

then the function $V(t, x(t))$ is nonincreasing in t, and

$$V(t, x(t)) \leqslant V(t_0, x_0), \qquad t \geqslant t_0.$$

COROLLARY 3.1.2. Assume that the hypotheses of Theorem 3.1.1 hold except that the condition (3.1.3) is to be satisfied only for $(t, x) \in J \times \Omega$, where

$$\Omega = [x \in R^n : r(t) < V(t, x) < r(t) + \epsilon_0, t \geqslant t_0],$$

ϵ_0 being some positive number. Then, the conclusion of Theorem 3.1.1 is true.

Proof. We choose $u_0 = V(t_0, x_0)$ and proceed as in the proof of Theorem 3.1.1 to obtain

$$D^+m(t) \leqslant g(t, m(t)), \qquad t \in Z,$$

Z being the set

$$Z = [t \in J : r(t) < m(t) < r(t) + \epsilon_0].$$

Theorem 1.4.2 now assures the stated result.

Sometimes, the following variants of Theorem 3.1.1 are more useful in applications.

THEOREM 3.1.2. Assume that the hypotheses of Theorem 3.1.1 hold except that the inequality (3.1.3) is replaced by

$$A(t)D^+V(t, x) + V(t, x)D^+A(t) \leqslant g(t, V(t, x)A(t)), \tag{3.1.7}$$

for $(t, x) \in J \times R^n$, where the function $A(t) > 0$ is continuous for $t \in J$, and

$$D^+A(t) = \limsup_{h \to 0^+} \frac{1}{h} [A(t + h) - A(t)].$$

Then

$$V(t_0, x_0)A(t_0) \leqslant u_0 \tag{3.1.8}$$

implies

$$V(t, x(t))A(t) \leqslant r(t), \qquad t \geqslant t_0. \tag{3.1.9}$$

Proof. Defining

$$V_1(t, x) = V(t, x)A(t),$$

it is easy to show that $V_1(t, x)$ satisfies the assumptions of Theorem 3.1.1.

For, if $h > 0$ is sufficiently small,

$$\begin{aligned} V_1(t+h, x+hf(t,x)) &- V_1(t,x) \\ &= V(t+h, x+hf(t,x))[A(t+h) - A(t)] \\ &\quad + A(t)[V(t+h, x+hf(t,x)) - V(t,x)], \end{aligned}$$

and therefore, using the assumption (3.1.7), we get

$$D^+V_1(t,x) \leqslant g(t, V_1(t,x)).$$

The estimate (3.1.9) follows immediately from Theorem 3.1.1.

THEOREM 3.1.3. Let the hypotheses of Theorem 3.1.1 hold except that, instead of the inequality (3.1.3), we now assume

$$D^+V(t,x) + \phi(\|x\|) \leqslant g(t, V(t,x)), \qquad (t,x) \in J \times R^n, \tag{3.1.10}$$

where $\phi(u) \geqslant 0$ is continuous for $u \geqslant 0$, $\phi(0) = 0$, and $\phi(u)$ is strictly increasing in u. Suppose further that $g(t,u)$ is nondecreasing in u for each $t \in J$. Then (3.1.5) implies that

$$V(t, x(t)) + \int_{t_0}^{t} \phi(\|x(s)\|)\, ds \leqslant r(t), \qquad t \geqslant t_0 . \tag{3.1.11}$$

Proof. Consider the function

$$m(t) = V(t, x(t)) + \int_{t_0}^{t} \phi(\|x(s)\|)\, ds,$$

so that $V(t, x(t)) \leqslant m(t)$. The monotonic character of $g(t,u)$ in u, together with the condition (3.1.10), now yields

$$D^+m(t) \leqslant g(t, m(t)),$$

and the assertion (3.1.11) follows from Theorem 1.4.1.

DEFINITION 3.1.1. The function $V(t,x)$ is said to be *mildly unbounded* if, for every $T > 0$, $V(t,x) \to \infty$ as $\|x\| \to \infty$ uniformly for $t \in [0, T]$.

The mild unboundedness of $V(t,x)$ guarantees that, whenever $V(t, x(t))$ is finite, $\|x(t)\|$ is also finite. The assumption that the solutions $x(t)$ of (3.1.1) exist for all $t \geqslant t_0$, therefore, becomes superfluous, if $V(t,x)$ is further assumed to be mildly unbounded in the foregoing theorem. From this observation stems the following global existence theorem.

THEOREM 3.1.4. Let $V \in C[J \times R^n, R_+]$, $V(t, x)$ be mildly unbounded and locally Lipschitzian in x. Suppose that $g \in C[J \times R_+, R]$ and $r(t)$ is the maximal solution of (3.1.4) defined for $t \geqslant t_0$. Assume that (3.1.3) holds. Then, every solution $x(t)$ of (3.1.1) exists in the future, i.e., for all $t \geqslant t_0$, and (3.1.5) implies (3.1.6).

Proof. Suppose that the assertion that every solution $x(t)$ of (3.1.1) exists for all $t \geqslant t_0$ is false. Then, by Corollary 1.1.2, there exists a $t_1 > t_0$ such that $x(t)$ cannot be extended to the closed interval $t_0 \leqslant t \leqslant t_1$, which implies that there cannot exist an increasing sequence $\{t_n\} \to t_{1-}$ such that $\| x(t_n)\|$ is bounded. This, in its turn, yields that $\| x(t_n)\| \to \infty$ as $t_n \to t_{1-}$. On the basis of Theorem 3.1.1, it follows that (3.1.5) implies (3.1.6) for $t_0 \leqslant t \leqslant t_1$. By the assumption that $V(t, x)$ is mildly unbounded, the fact that $r(t)$ exists for all $t \geqslant t_0$, and (3.1.6), there arises a contradiction as $t_n \to t_{1-}$. Hence, the global existence of solutions $x(t)$ of (3.1.1) is proved, which, in turn, assures the estimate (3.1.6) for $t \geqslant t_0$ whenever (3.1.5) holds. The proof is complete.

3.2. Definitions

Let $x(t, t_0, x_0)$ be any solution of the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{3.2.1}$$

where $f \in C[J \times S_\rho, R^n]$, S_ρ being the set

$$S_\rho = [x \in R^n : \| x \| < \rho]. \tag{3.2.2}$$

Assume that $f(t, 0) = 0, t \in J$, so that $x = 0$ is a (trivial) solution of (3.2.1) through $(t_0, 0)$. We now list a few definitions concerning the stability of the trivial solution.

DEFINITION 3.2.1. The trivial solution $x = 0$ of (3.2.1) is

(S_1) *equistable* if, for each $\epsilon > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that the inequality

$$\| x_0 \| \leqslant \delta$$

implies

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0;$$

(S_2) *uniformly stable* if the δ in (S_1) is independent of t_0;

(S_3) *quasi-equi asymptotically stable* if, for each $\epsilon > 0$, $t_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that, for $t \geqslant t_0 + T$ and $\| x_0 \| < \delta_0$,

$$\| x(t, t_0, x_0) \| < \epsilon;$$

(S_4) *quasi uniformly asymptotically stable* if the numbers δ_0 and T in (S_3) are independent of t_0;

(S_5) *equi-asymptotically stable* if (S_1) and (S_3) hold simultaneously;

(S_6) *uniformly asymptotically stable* if (S_2) and (S_4) hold together;

(S_7) *quasi-equi asymptotically stable* if, for each $\epsilon > 0$, $\alpha > 0$, and $t_0 \in J$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that $\| x_0 \| \leqslant \alpha$ implies

$$\| x(t, t_0, x_0) \| < \epsilon, \qquad t \geqslant t_0 + T;$$

(S_8) *quasi uniformly asumptotically stable* if the T in (S_7) is independent of t_0;

(S_9) *completely stable* if (S_1) holds and (S_7) is verified for all α, $0 \leqslant \alpha < \infty$;

(S_{10}) *uniformly completely stable* if (S_2) holds and (S_8) is verified for all α, $0 \leqslant \alpha < \infty$;

(S_{11}) *unstable* if (S_1) fails to hold.

REMARK 3.2.1. Sometimes the notion of quasi-asymptotic stability may be relaxed somewhat as in (S_7) and (S_8). Clearly the ϵ, α given in the preceding definitions must be less than ρ of (3.2.2), and therefore the concepts (S_1)–(S_8) are of local nature. If, on the other hand, $\rho = \infty$, so that $S_\rho = R^n$, the corresponding concepts of stability would be of global character. These considerations lead to (S_9) and (S_{10}). We note further that the definitions (S_7) and (S_8) may hold even when $f(t, 0) \not\equiv 0$. In other words, the assumption about the existence of the trivial solution is not necessary.

In characterizing Lyapunov functions, it is convenient to introduce certain classes of monotone functions.

DEFINITION 3.2.2. A function $\phi(r)$ is said to belong to the class $\mathscr{K}$ if $\phi \in C[[0, \rho), R_+]$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotone increasing in r.

DEFINITION 3.2.3. A function $\sigma(t)$ is said to belong to the class $\mathscr{L}$ if $\sigma \in C[J, R_+]$, $\sigma(t)$ is monotone decreasing in t, and $\sigma(t) \to 0$ as $t \to \infty$.

DEFINITION 3.2.4. A function $\phi(t, r)$ is said to belong to the class $\mathscr{K}\mathscr{K}$ if $\phi \in C[J \times [0, \rho), R_+]$, $\phi \in \mathscr{K}$ for each $t \in J$, and ϕ is monotone increasing in t for each $r > 0$ and $\phi(t, r) \to \infty$ as $t \to \infty$ for each $r > 0$.

DEFINITION 3.2.5. A function $V(t, x)$ with $V(t, 0) \equiv 0$ is said to be *positive definite* (*negative definite*) if there exists a function $\phi(r) \in \mathscr{K}$ such that the relation

$$V(t, x) \geqslant \phi(\| x \|), \qquad (\leqslant -\phi(\| x \|))$$

is satisfied for $(t, x) \in J \times S_\rho$.

DEFINITION 3.2.6. A function $V(t, x)$ with $V(t, 0) \equiv 0$ is said to be *strongly positive definite* if there exists a function $\phi(t, r) \in \mathscr{K}\mathscr{K}$ such that

$$V(t, x) \geqslant \phi(t, \| x \|), \qquad (t, x) \in J \times S_\rho .$$

DEFINITION 3.2.7. A function $V(t, x) \geqslant 0$ is said to be *decrescent* if a function $\phi(r) \in \mathscr{K}$ exists such that

$$V(t, x) \leqslant \phi(\| x \|), \qquad (t, x) \in J \times S_\rho .$$

To use the second method of Lyapunov, which attempts to make statements about the stability properties directly by using suitable functions, we need to study the scalar differential equation

$$u' = g(t, u), \qquad u(t_0) = u_0 \geqslant 0, \qquad t_0 \geqslant 0, \tag{3.2.3}$$

where $g \in C[J \times R_+, R]$. We suppose that $g(t, 0) \equiv 0$ so that $u = 0$ is a solution of (3.2.3) through $(t_0, 0)$. Furthermore, this assumption also implies that the solutions $u(t) = u(t, t_0, u_0)$ of (3.2.3) are non-negative for $t \geqslant t_0$ so as to assure that $g(t, u(t))$ is defined.

Corresponding to the stability definitions (S_1)–(S_8), we designate by (S_1^*)–(S_8^*) the concepts concerning the stability of the solution $u = 0$ of (3.2.3).

DEFINITION 3.2.8. The trivial solution $u = 0$ of (3.2.3) is said to be (S_1^*) *equistable* if, for each $\epsilon > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that

$$u(t, t_0, u_0) < \epsilon, \qquad t \geqslant t_0 ,$$

provided

$$u_0 \leqslant \delta.$$

The definitions (S_2^*)–(S_8^*) may be formulated similarly. Notice as before that, for the concepts (S_7^*) and (S_8^*) to hold, the assumption $g(t, 0) \equiv 0$ is not necessary.

3.3. Stability

We begin with the following stability criteria, recalling the definition of the function $D^+V(t, x)$ given in (3.1.2).

THEOREM 3.3.1. Assume that there exist functions $V(t, x)$ and $g(t, u)$ satisfying the following conditions:

(i) $g \in C[J \times R_+, R]$ and $g(t, 0) \equiv 0$.

(ii) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$, and $V(t, x)$ is positive definite and locally Lipschitzian in x.

(iii) for $(t, x) \in J \times S_\rho$,

$$D^+V(t, x) \leqslant g(t, V(t, x)).$$

Then, the equistability of the trivial solution of (3.2.3) implies the equistability of the trivial solution of the system (3.2.1).

Proof. By assumption, a function $b(r)$ of class $\mathscr{K}$ exists such that

$$V(t, x) \geqslant b(\| x \|), \qquad (t, x) \in J \times S_\rho . \tag{3.3.1}$$

Let $0 < \epsilon < \rho$ and $t_0 \in J$ be given. Since the solution $u = 0$ is equistable, given $b(\epsilon) > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, such that $u_0 \leqslant \delta$ implies

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 . \tag{3.3.2}$$

Choose $u_0 = V(t_0, x_0)$. Since $V(t, x)$ is continuous and $V(t, 0) \equiv 0$, it is possible to find a positive function $\delta_1 = \delta_1(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, satisfying the inequalities

$$\| x_0 \| \leqslant \delta_1 , \qquad V(t_0, x_0) \leqslant \delta \tag{3.3.3}$$

simultaneously. We claim that, if $\| x_0 \| \leqslant \delta_1$,

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0 .$$

Suppose that this is not true. Then, there would exist a solution $x(t) = x(t, t_0, x_0)$ with $\| x_0 \| \leqslant \delta_1$ and a $t_1 > t_0$ such that

$$\| x(t_1)\| = \epsilon, \qquad \| x(t)\| \leqslant \epsilon, \qquad t \in [t_0, t_1],$$

so that

$$V(t_1, x(t_1)) \geqslant b(\epsilon) \tag{3.3.4}$$

because of (3.3.1). This means that $\| x(t)\| < \rho$ for $t \in [t_0, t_1]$, and hence the choice $u_0 = V(t_0, x_0)$ and condition (iii) give, as a consequence of Theorem 3.1.1, the estimate

$$V(t, x(t)) \leqslant r(t, t_0, u_0), \qquad t \in [t_0, t_1], \tag{3.3.5}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.2.3). The relations (3.3.2), (3.3.4), and (3.3.5) lead to the contradiction

$$b(\epsilon) \leqslant V(t_1, x(t_1)) \leqslant r(t_1, t_0, u_0) < b(\epsilon),$$

proving (S_1). The proof of the theorem is complete.

COROLLARY 3.3.1. The function $g(t, u) = \lambda(t)\,\phi(u)$, where $\lambda(t)$ is continuous on J, $\phi(u) \geqslant 0$ is continuous for $u \geqslant 0$, $\phi(u) > 0$ for $u > 0$, is admissible in Theorem 3.3.1 provided

$$-\int_0^{u_0} \frac{ds}{\phi(s)} \leqslant \int_{t_0}^{t} \lambda(s)\, ds < \int_{u_0}^{\infty} \frac{ds}{\phi(s)}, \tag{3.3.6}$$

for some $u_0 > 0$, every $t_0 \geqslant 0$ and $t_0 \leqslant t < \infty$.

Proof. All that we have to verify is that the solution $u = 0$ of (3.2.3) is equistable. Define

$$J(u) = \int_0^{u} \frac{ds}{\phi(s)} \qquad \text{if} \quad \int_0^{u} \frac{ds}{\phi(s)} < \infty;$$

otherwise, $J(u) = \int_\epsilon^u ds/\phi(s)$, for sufficiently small $\epsilon > 0$. If $\int_0^\infty ds/\phi(s) = R \leqslant \infty$, then $J(u)$ is a monotone function mapping the interval $[0, \infty)$ homeomorphically onto the interval $[0, R)$. The solution $u(t, t_0, u_0)$ of (3.2.3) is given by

$$J(u(t, t_0, u_0)) = J(u_0) + \int_{t_0}^{t} \lambda(s)\, ds \tag{3.3.7}$$

as long as

$$0 \leqslant J(u_0) + \int_{t_0}^{t} \lambda(s)\, ds < R,$$

or

$$-\int_0^{u_0} \frac{ds}{\phi(s)} \leqslant \int_{t_0}^{t} \lambda(s)\, ds < \int_{u_0}^{\infty} \frac{ds}{\phi(s)}.$$

Thus, from (3.3.7) and the fact that J is a homeomorphism, it is easy to see that $u = 0$ of (3.2.3) is equistable when (3.3.6) holds. This proves the corollary.

THEOREM 3.3.2. Under the assumptions of Theorem 3.3.1, the uniform stability of the solution $u = 0$ of (3.2.3) also implies the equistability of the trivial solution of (3.2.1).

Proof. The proof follows from the proof of Theorem 3.3.1. In this case, although δ is independent of t_0, the relation (3.3.3) shows that δ_1 is not independent of t_0. Consequently, one gets only the equistability of the trivial solution of (3.2.1).

COROLLARY 3.3.2. Assume that there exists a function $V(t, x)$ verifying the following conditions:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ is positive definite and locally Lipschitzian in x.

(ii) $D^+ V(t, x) \leqslant 0$, $(t, x) \in J \times S_\rho$.

Then, the trivial solution of (3.2.1) is equistable.

It is important to note that, when (ii) holds, the scalar differential equation (3.2.3) reduces to

$$u' = 0, \qquad u(t_0) = u_0 \geqslant 0, \qquad t_0 \geqslant 0,$$

and as a result (S_2^*) is satisfied. Thus, Corollary 3.3.2 follows from Theorem 3.3.2.

THEOREM 3.3.3. In addition to the hypotheses of Theorem 3.3.1, assume that $V(t, x)$ is decrescent. Then, the equistability of the null solution of (3.2.3) assures the equistability of the solution $x = 0$ of (3.2.1).

Proof. Since $V(t, x)$ is decrescent, there exists a function $a(r) \in \mathscr{K}$ such that

$$V(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times S_\rho .$$

We follow the proof of Theorem 3.3.1 except that we choose $u_0 = a(\| x_0 \|)$. By assumption, (S_1^*) holds, and therefore $\delta = \delta(t_0, \epsilon)$ depends on t_0. As $a(r) \in \mathscr{K}$, the existence of a positive function $\delta_1 = \delta_1(t_0, \epsilon)$ satisfying the inequalities

$$\| x_0 \| \leqslant \delta_1, \qquad a(\| x_0 \|) \leqslant \delta \tag{3.3.8}$$

simultaneously is clear. The rest of the proof is very much the same.

COROLLARY 3.3.3. The function $g(t, u) = \lambda(t)u$, where $\lambda(t)$ is continuous on J and

$$\int_{t_0}^{\infty} \lambda(s)\, ds < \infty$$

for every $t_0 \geqslant 0$, is admissible in Theorem 3.3.3.

THEOREM 3.3.4. Let the hypotheses of Theorem 3.3.1 hold. Assume further that $V(t, x)$ is decrescent. Then, the uniform stability of the solution $u = 0$ of (3.2.3) guarantees the uniform stability of the trivial solution of (3.2.1).

Proof. Following the proof of Theorem 3.3.3, it is easy to see that δ_1 does not depend on t_0. For, by assymption of the uniform stability of the null solution of (3.2.3), δ is independent of t_0, and (3.3.8) shows that δ_1 is also independent of t_0.

COROLLARY 3.3.4. The function $g(t, u) = \lambda(t)\,\phi(u)$ defined in Corollary 3.3.1 is admissible in Theorem 3.3.4 if (3.3.6) holds and if either $\int_{u_0}^{\infty} ds/\phi(s) < \infty$ or (i) $\int_{u_0}^{\infty} ds/\phi(s) = \infty$ and (ii) $\int_{t_0}^{t} \lambda(s)\, ds$ is bounded above for every t, $t_0 \leqslant t < \infty$ uniformly in t_0.

COROLLARY 3.3.5. Assume that there exists a function $V(t, x)$ fulfilling the following assumptions:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is positive definite and decrescent, and $V(t, x)$ is locally Lipschitzian in x.

(ii) $D^+V(t, x) \leqslant 0$, $(t, x) \in J \times S_\rho$.

Then, the trivial solution of (3.2.1) is uniformly stable.

The definition of uniform stability of the solution $x = 0$ given in (S_2) can also be formulated by means of a monotone function, as can be seen by the following

THEOREM 3.3.5. The trivial solution of (3.2.1) is uniformly stable if and only if there exists a function $a(r) \in \mathscr{K}$ verifying the estimate

$$\| x(t, t_0, x_0)\| \leqslant a(\| x_0 \|), \qquad t \geqslant t_0, \tag{3.3.9}$$

for $\| x_0 \| < \rho$.

Proof. The sufficiency of the condition is immediately clear. To prove the necessity, consider, for a given $\epsilon > 0$, the least upper bound of all positive functions $\delta(\epsilon)$, and designate it by $\bar{\delta} = \bar{\delta}(\epsilon)$. Then $\| x_0 \| \leqslant \bar{\delta}$

implies $\| x(t, t_0, x_0)\| \leqslant \epsilon$ for $t \geqslant t_0$, and, if $\delta_1 > \bar{\delta}$, there exists at least one x_0 such that, for $\| x_0 \| < \delta_1$, $\| x(t, t_0, x_0)\|$ exceeds the value ϵ at some time t. Clearly, the function $\bar{\delta}(\epsilon)$ is positive for $\epsilon > 0$; it is nondecreasing and tends to zero as $\epsilon \to 0$; and it may be discontinuous. We now choose a continuous, monotonically increasing function $\delta^*(\epsilon)$ satisfying $\delta^*(\epsilon) \leqslant \bar{\delta}(\epsilon)$. Then, the inverse function $a(r) = \delta^{*-1}(r)$ satisfies (3.3.9). The proof is complete.

We shall now prove a result that gives sufficient conditions for unstability of the solution $x = 0$ of (3.2.1).

THEOREM 3.3.6. Let there exist functions $V(t, x)$ and $g(t, u)$ satisfying the following properties:

(i) $V \in C[\bar{G}, R_+]$, $V(t, x)$ is locally Lipschitzian in x on $\bar{G}$, $V(t, x) = 0$ for all $(t, x) \in \bar{G} - G$, and $V(t, x)$ is positive and bounded on G, where $G \subset J \times S_\rho$ is some open set such that G has at least one boundary point $(T, 0)$, $T > 0$.

(ii) $g \in C[J \times R_+, R_+]$, and

$$D^+V(t, x) \geqslant g(t, V(t, x)) \geqslant 0, \qquad (t, x) \in G.$$

(iii) For $t_0 > T$, the solutions $u(t, t_0, u_0)$ of (3.2.3), for arbitrarily small $u_0 > 0$, are either unbounded or indeterminate, for $t \geqslant t_0$.

Then the trivial solution of (3.2.1) is unstable.

Proof. There exists a point $(t_0, x_0) \in G$, $x_0 \neq 0$, in the vicinity of $(T, 0)$. Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.2.1). Then, the Lipschitzian nature of $V(t, x)$ and condition (ii) yield

$$V(t, x(t)) \geqslant V(t_0, x_0) = u_0 > 0, \tag{3.3.10}$$

for all $t \geqslant 0$, for which $(t, x(t)) \in G$. Since $V(t, x) = 0$ for all $(t, x) \in \bar{G} - G$, it follows from (3.3.10) that $(t, x(t)) \in G$ for $t \geqslant t_0$. Moreover, we also have

$$D^+V(t, x(t)) \geqslant g(t, V(t, x(t)),$$

which, in view of Remark 1.4.1, implies that

$$V(t, x(t)) \geqslant \rho(t, t_0, u_0), \qquad t \geqslant t_0, \tag{3.3.11}$$

where $\rho(t, t_0, u_0)$ is the minimal solution of (3.2.3). Since $V(t, x)$ is bounded by assumption, the estimate (3.3.11) leads to an absurdity,

if we assume the trivial solution of (3.2.1) is stable. This proves the theorem.

THEOREM 3.3.7. Let $f \in C[(-\infty, \infty) \times S_\rho, R^n]$ and $f(t, x)$ be periodic in t with a period ω. Then, under the hypotheses of Theorem 3.3.1, the trivial solution of (3.2.1) is equistable for $t_0 \in (-\infty, \infty)$.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in (-\infty, \infty)$ be given. It is possible to choose an integer k such that $t_0 + k\omega \geqslant 0$. Since the solution $u = 0$ of (3.2.3) is equistable, given $b(\epsilon) > 0$, $t_0 + k\omega \geqslant 0$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that $u_0 \leqslant \delta$ implies

$$u(t + k\omega, t_0 + k\omega, u_0) < b(\epsilon), \qquad t \geqslant t_0, \tag{3.3.12}$$

$u(t + k\omega, t_0 + k\omega, u_0)$ being any solution of

$$u' = g(t + k\omega, u), \qquad u(t_0 + k\omega) = u_0. \tag{3.3.13}$$

We choose $u_0 = V(t_0 + k\omega, x_0)$ and obtain $\delta_1 = \delta_1(t_0, \epsilon)$ satisfying

$$\| x_0 \| \leqslant \delta_1, \qquad V(t_0 + k\omega, x_0) \leqslant \delta$$

simultaneously, as in the proof of Theorem 3.3.1. With this δ_1, the equistability of the trivial solution of (3.2.1) follows. Supposing this were false, there would exist a $t_1 > t_0$ such that

$$\| x(t_1) \| = \epsilon, \qquad \| x(t) \| \leqslant \epsilon, \qquad t \in [t_0, t_1] \tag{3.3.14}$$

for some solution $x(t)$ of (3.2.1) such that $\| x_0 \| \leqslant \delta_1$. Consider the function

$$m(t) = V(t + k\omega, x(t)),$$

and, for small $h > 0$,

$$\begin{aligned} m(t + h) - m(t) \leqslant{} & Lh \| f(t, x(t)) - f(t + k\omega, x(t)) \| \\ & + L \| \epsilon(h) \| + [V(t + k\omega + h, x(t) + hf(t + k\omega, x(t))) \\ & - V(t + k\omega, x(t))], \end{aligned}$$

where L is the Lipschitz constant and $\epsilon(h)/h \to 0$ as $h \to 0$. The periodicity of $f(t, x)$ and condition (ii) of Theorem 3.3.1 give the inequality

$$D^+ m(t) \leqslant g(t + k\omega, m(t)), \qquad t \in [t_0, t_1],$$

which implies the estimate, by Theorem 1.4.1,

$$V(t + k\omega, x(t)) \leqslant r(t + k\omega, t_0 + k\omega, u_0), \qquad t \in [t_0, t_1], \tag{3.3.15}$$

$r(t + k\omega, t_0 + k\omega, u_0)$ being the maximal solution of (3.3.13). Thus,

$$b(\epsilon) \leqslant V(t_1 + k\omega, x(t_1)) \leqslant r(t_1 + k\omega, t_0 + k\omega, u_0) < b(\epsilon),$$

using relations (3.3.1), (3.3.12), (3.3.14), and (3.3.15). This contradiction proves the assertion of the theorem.

If, in particular, the function $f(t, x)$ of system (3.2.1) is known to be periodic in t or autonomous and is smooth enough to assure uniqueness of solutions, it is possible to infer more information about the stability of the null solution of (3.2.1). To this effect, we have

THEOREM 3.3.8. Let $f \in C[(-\infty, \infty) \times S_\rho, R^n]$, $f(t, x)$ be periodic in t with a period ω, and the system (3.2.1) admit unique solutions. Then, the stability of the trivial solution of (3.2.1) is necessarily uniform.

Proof. By the periodicity of $f(t, x)$ in t, it follows that, if $x(t, t_0, x_0)$ is a solution of (3.2.1), then $x(t + \omega, t_0, x_0)$ is also a solution. Furthermore, the uniqueness of solutions shows that, for any interger k, $x(t \pm k\omega, t_0 \pm k\omega, x_0) = x(t, t_0, x_0)$.

For each fixed t_0, $t_0 \in (-\infty, \infty)$, let

$$\delta(t_0) = \sup_{0<\epsilon<\rho} \delta(t_0, \epsilon).$$

Since $\delta(t_0, \epsilon)$ is continuous in t_0 for each ϵ, if we let $\delta_0 = \inf_{0 \leqslant t_0 \leqslant \omega} \delta(t_0)$, it is clear that $\delta_0 > 0$. For $\sigma < \delta < \delta_\sigma$, we define

$$\epsilon(\delta) = \sup_{\sigma \geqslant 0} \| x(t_0 + \sigma, t_0, x_0)\|.$$

Notice that $\epsilon(\delta)$ is a monotone increasing function of δ, and hence, if $\delta(\epsilon)$ is the inverse function of $\epsilon(\delta)$, we have

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0,$$

for every $t_0 \in [0, \omega]$, provided $\| x_0 \| \leqslant \delta(\epsilon)$. Let t_0 be arbitrary. Then we can choose an integer k such that

$$k\omega \leqslant t_0 \leqslant (k + 1)\omega \qquad \text{or} \qquad -(k + 1)\omega \leqslant t_0 \leqslant -k\omega.$$

Hence, either $0 \leqslant t_0 - k\omega \leqslant \omega$ or $0 \leqslant t_0 + (k + 1)\omega \leqslant \omega$.

Consequently, if $\| x_0 \| \leqslant \delta(\epsilon)$, either

$$\| x(t, t_0, x_0)\| = \| x(t - k\omega, t_0 - k\omega, x_0)\| < \epsilon$$

or

$$\| x(t, t_0, x_0)\| = \| x(t + (k + 1)\omega, t_0 + (k + 1)\omega, x_0)\| < \epsilon,$$

for $t \geqslant t_0$, and the uniformity of the stability is evident.

3.4. Asymptotic stability

We present, in this section, a number of results concerning the asymptotic stability of the solution $x = 0$ of the system (3.2.1).

THEOREM 3.4.1. Let there exist functions $V(t, x)$ and $g(t, u)$ fulfilling the following assumptions:

(i) $g \in C[J \times R_+, R]$ and $g(t, 0) \equiv 0$.

(ii) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$, and $V(t, x)$ is positive definite and locally Lipschitzian in x.

(iii) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times S_\rho$.

Then, the equi-asymptotic stability of the solution $u = 0$ of (3.2.3) assures the equi-asymptotic stability of the trivial solution of (3.2.1).

Proof. Let (S_5^*) hold. Then, by definition, (S_1^*) and (S_3^*) are satisfied, and therefore only quasi-equi asymptotic stability needs to be proved, since Theorem 3.3.1 guarantees (S_1).

To prove (S_3), let $\epsilon > 0$, $t_0 \in J$ be given. It then follows from (S_3^*) that, given $b(\epsilon) > 0$, $t_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 + T, \tag{3.4.1}$$

whenever $u_0 \leqslant \delta_0$. Choose $u_0 = V(t_0, x_0)$. Since $V(t, x)$ is continuous and $V(t, 0) \equiv 0$, we can find, as in the proof of Theorem 3.3.1, a positive number $\hat{\delta}_0 = \hat{\delta}_0(t_0)$ satisfying the inequalities

$$\| x_0 \| \leqslant \hat{\delta}_0, \qquad V(t_0, x_0) \leqslant \delta_0 \tag{3.4.2}$$

together. We have also, in view of (S_1), the inequality (3.3.5) holding for $t \geqslant t_0$. Suppose, now, that there is a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$ and $t_k \to \infty$ as $k \to \infty$ such that $\| x(t_k, t_0, x_0)\| \geqslant \epsilon$, where $x(t, t_0, x_0)$ is some solution of (3.2.1) starting in $\| x_0 \| \leqslant \hat{\delta}_0$. This leads to the contradiction

$$b(\epsilon) \leqslant V(t_k, x(t_k, t_0, x_0)) \leqslant r(t_k, t_0, u_0) < b(\epsilon),$$

because of (3.3.1), (3.3.5), and (3.4.1). Thus, (S_3) is proved, which implies the equi-asymptotic stability of the solution $x = 0$ of (3.2.1).

REMARK 3.4.1. It is possible that $\hat{\delta}_0$ obtained in (3.4.2) may be such that $\hat{\delta}_0 > \delta_0^*$ where $\delta_0^* = \delta(t_0, \rho)$, in which case the inequality (3.3.5)

need not hold for all $t \geqslant t_0$ whenever $\| x_0 \| < \delta_0$. To avoid this situation, it is necessary to redefine $\delta_0 = \min[\delta_0, \delta_0^*]$ in the foregoing proof.

COROLLARY 3.4.1. The function $g(t, u) = \lambda(t)\,\phi(u)$, where $\lambda(t)$ is continuous on J, $\phi(u) \geqslant 0$ is continuous for $u \geqslant 0$, $\phi(u) > 0$ for $u > 0$, is admissible in Theorem 3.4.1 provided that there exists a T, $t_0 \leqslant T \leqslant \infty$, verifying the relation

$$\int_{t_0}^{T} \lambda(s)\, ds = -\int_0^{u_0} \frac{ds}{\phi(s)}.$$

THEOREM 3.4.2. Under the assumptions of Theorem 3.4.1, the uniform asymptotic stability of the trivial solution of (3.2.3) also implies the equi-asymptotic stability of the solution $x = 0$ of (3.2.1).

Proof. The proof runs parallel to the proof of Theorem 3.4.1. First of all, following the proof of Theorem 3.3.2, we note that δ_1 is not independent of t_0 although δ is. Similarly, δ_0 occuring in (3.4.2) is also not independent of t_0, even though δ_0 is.

COROLLARY 3.4.2. The conclusion of Theorem 3.4.2 remains true if the function $g(t, u) = -\phi(u)$, where $\phi(u) \in \mathscr{K}$.

Proof. We shall show that (S_6^*) holds. Consider the scalar differential equation (3.2.3), which now takes the form

$$u' = -\phi(u), \qquad u(t_0) = u_0 \geqslant 0, \qquad t_0 \geqslant 0,$$

whose solutions are easily seen to be

$$u(t, t_0, u_0) = J^{-1}[J(u_0) - (t - t_0)], \qquad t \geqslant t_0, \tag{3.4.3}$$

where

$$J(u) - J(u_0) = \int_{u_0}^{u} \frac{ds}{\phi(s)},$$

and J^{-1} is the inverse function of J. Given any $\epsilon > 0$, we first observe that (S_2) holds with $\delta(\epsilon) = \epsilon$. Furthermore, we can also conclude from (3.4.3) that (S_4^*) is satisfied with $\delta_0 = \rho$ and $T(\epsilon) = J(\rho) - J(\epsilon)$, for any $\epsilon < \rho$. Thus, it follows that the solution $u = 0$ of (3.2.3) is uniformly asymptotically stable.

THEOREM 3.4.3. Let the hypotheses of Theorem 3.4.1 hold. Suppose further that $V(t, x)$ is decrescent. Then, if the solution $u = 0$ of (3.2.3) is equi-asymptotically stable, the trivial solution of (3.2.1) is equi-asymptotically stable.

Proof. The equistability of the solution $x = 0$ is immediate from Theorem 3.3.3. To prove (S_3), we follow the proof of Theorem 3.4.1 and choose $u_0 = a(\| x_0 \|)$. Then, instead of (3.4.2), we have the inequalities

$$\| x_0 \| \leqslant \hat{\delta}_0, \qquad a(\| x_0 \|) \leqslant \delta_0 \tag{3.4.4}$$

holding simultaneously. The rest of the proof runs almost similar. The fact that $\hat{\delta}_0$ and T are not independent of t_0 shows that (S_3) holds. This proves (S_5), and the proof is complete.

COROLLARY 3.4.3. The function $g(t, u) = \lambda(t)u$, where $\lambda(t)$ is continuous on J and

$$\int_{t_0}^{\infty} \lambda(s)\, ds \to -\infty$$

for every $t_0 \geqslant 0$, is admissible in Theorem 3.4.3.

THEOREM 3.4.4. Let the hypotheses of Theorem 3.4.1 hold, and let $V(t, x)$ be decrescent. Then, uniform asymptotic stability of the solution $u = 0$ of (3.2.3) guarantees likewise the uniform asymptotic stability of the null solution of (3.2.1).

Proof. Since uniform stability of the solution $x = 0$ follows from Theorem 3.3.4, it remains to be shown that (S_4) holds. To do this, we follow the proof of Theorem 3.4.3 and observe that, in view of (3.4.4), $\hat{\delta}_0$ is independent of t_0. That the number $T(\epsilon)$ depends only on ϵ follows from the condition (S_4^*). Hence, (S_4) is satisfied, which, in its turn, proves (S_6).

THEOREM 3.4.5. Assume that there exist functions $V(t, x)$ and $g(t, u)$ obeying the following conditions:

(i) $g \in C[J \times R_+, R]$ and $g(t, 0) = 0$, $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) = 0$, $t \in J$, and $V(t, x)$ is strongly positive definite and locally Lipschitzian in x.

(iii) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times S_\rho$.

Then, if the solution $u = 0$ of (3.2.3) is equistable, the trivial solution of (3.2.1) is equi-asymptotically stable.

Proof. By assumption (ii), $V(t, x)$ is strongly positive definite, which implies that there exists a function $b(t, u) \in \mathscr{K}\mathscr{K}$ such that

$$V(t, x) \geqslant b(t, \| x \|), \qquad (t, x) \in J \times S_\rho. \tag{3.4.5}$$

Define $b_1(u) = b(0, u)$. Then

$$V(t, x) \geqslant b_1(\| x \|), \qquad (t, x) \in J \times S_\rho . \tag{3.4.6}$$

Let $0 < \eta < \rho$ be given and $t_0 \in J$. Since (S_1^*) holds, given $b_1(\eta) > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \eta)$, which is continuous in t_0 for each η such that $u_0 \leqslant \delta$ implies

$$u(t, t_0, u_0) < b_1(\eta), \qquad t \geqslant t_0 . \tag{3.4.7}$$

Choosing $u_0 = V(t_0, x_0)$, we can find a positive function $\delta_1 = \delta_1(t_0, \eta)$, as in the proof of Theorem 3.3.1, such that the inequalities

$$\| x_0 \| \leqslant \delta_1 , \qquad V(t_0, x_0) \leqslant \delta \tag{3.4.8}$$

hold together. Furthermore, by Theorem 3.3.1, we see that the solution $x = 0$ of (3.2.1) is equistable, using (3.4.6). Let η be fixed, and let δ_0 denote the number $\delta_1(t_0, \eta)$. To prove (S_3), let $0 < \epsilon < \eta$, $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \delta_0$. Since $b(t, u) \in \mathscr{K}\mathscr{K}$, there exists a $T = T(t_0, \epsilon)$ satisfying the relation

$$b(t, \epsilon) > \sup_{\|x_0\| \leqslant \delta_0} V(t_0, x_0), \qquad t \geqslant t_0 + T. \tag{3.4.9}$$

If $\{t_k\}$ is a sequence such that $t_k \geqslant t_0 + T$, $t_k \to \infty$ as $k \to \infty$, such that $\| x(t_k, t_0, x_0)\| \geqslant \epsilon$ whenever $\| x_0 \| \leqslant \delta_0$, it would follow from (3.4.5), (3.4.7), (3.4.9), and Theorem 3.1.1 that

$$b(t_k, \epsilon) \leqslant V(t_k, x(t_k, t_0, x_0)) \leqslant r(t_k, t_0, u_0) < b_1(\eta). \tag{3.4.10}$$

This is a contradiction, since $b(t_k, \epsilon) \to \infty$ as $t_k \to \infty$. Thus, (S_1) and (S_3) hold simultaneously, and the theorem is proved.

THEOREM 3.4.6. Under the assumptions of Theorem 3.4.5, the uniform stability of the solution $u = 0$ of (3.2.3) also implies the equi-asymptotic stability of the solution $x = 0$ of (3.2.1).

Proof. By assumption, δ is independent of t_0 in the foregoing proof. However, $\delta_0 = \delta_1$ is not independent of t_0 because of (3.4.8). Moreover, T also depends on t_0, in view of (3.4.9), and thus (S_5) holds.

COROLLARY 3.4.4. The function $g(t, u) \equiv 0$ is admissible in Theorem 3.4.6.

THEOREM 3.4.7. Assume that there exist functions $V(t, x)$, $g(t, u)$, and $A(t)$ satisfying the following properties:

(i) $A(t) > 0$ is continuous for $t \in J$ and $A(t) \to \infty$ as $t \to \infty$.

(ii) $g \in C[J \times R_+, R]$ and $g(t, 0) = 0$, $t \in J$.

(iii) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) = 0$, $t \in J$, and $V(t, x)$ is positive definite and locally Lipschitzian in x.

(iv) $A(t)D^+V(t, x) + V(t, x)D^+A(t) \leqslant g(t, V(t, x)A(t))$, $(t, x) \in J \times S_\rho$.

Then, if the null solution of (3.2.3) is equistable, the solution $x = 0$ of (3.2.1) is equi-asymptotically stable.

Proof. Let $0 < \eta < \rho$, $t_0 \in J$ be given, and let $\sigma = \min_{t \in J} A(t)$. By assumption (i), $\sigma > 0$. Define $\eta^* = \sigma b(\eta)$, where $b(u) \in \mathscr{K}$ is the same function as in (3.3.1), obtained because of the positive definiteness of $V(t, x)$. Assume that (S_1^*) holds. Then, given $\eta^* > 0$, $t_0 \in J$, there exists a $\delta = \delta(t_0, \eta)$ that is continuous in t_0 for each η such that

$$u(t, t_0, u_0) < \eta^*, \qquad t \geqslant t_0, \tag{3.4.11}$$

if $u_0 \leqslant \delta$. Choose $u_0 = A(t_0)\, V(t_0, x_0)$. Arguing, as in Theorem 3.3.1, we conclude the existence of a positive function $\hat{\delta} = \hat{\delta}(t_0, \eta)$ such that

$$\| x_0 \| \leqslant \hat{\delta}, \qquad A(t_0)V(t_0, x_0) \leqslant \delta \tag{3.4.12}$$

hold jointly. With this $\hat{\delta}$, condition (S_1) holds. For otherwise we are led to a contradiction, as in the proof of Theorem 3.3.1,

$$\sigma b(\eta) \leqslant A(t_1)V(t_1, x(t_1, t_0, x_0)) \leqslant r(t_1, t_0, u_0) < \eta^*,$$

because of the definition of $\eta^* = \sigma b(\eta)$ and the application of Theorem 3.1.2.

For a fixed $\eta < \rho$, designate by $\delta_0 = \delta_0(t_0)$ the number $\hat{\delta}(t_0, \eta)$. Let now $0 < \epsilon < \eta$, $t_0 \in J$ be given, and let $\| x_0 \| < \delta_0$. Since $A(t) \to \infty$ as $t \to \infty$, there exists a number $T = T(t_0, \epsilon)$ such that

$$b(\epsilon)A(t) > \eta^*, \qquad t \geqslant t_0 + T. \tag{3.4.13}$$

The choice $u_0 = A(t_0)\, V(t_0, x_0)$, the positive definiteness of $V(t, x)$, and Theorem 3.1.2 give

$$A(t)b(\| x(t, t_0, x_0)\|) \leqslant A(t)V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0) \tag{3.4.14}$$

for $t \geqslant t_0$, where $x(t, t_0, x_0)$ is any solution of (3.2.1) such that $\| x_0 \| < \delta_0$. If there exists a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$, $t_k \to \infty$ as

$k \to \infty$ such that $\| x(t_k, t_0, x_0)\| \geqslant \epsilon$ for some solution satisfying $\| x_0 \| < \delta_0$, we obtain from (3.4.11) and (3.4.14) the inequality

$$A(t_k)b(\epsilon) < \eta^*,$$

which contradicts (3.4.13). This proves that (S_3) holds, and consequently the solution $x = 0$ is equi-asymptotically stable. The theorem is proved.

THEOREM 3.4.8. Under the hypotheses of Theorem 3.4.7, the uniform stability of the solution $u = 0$ of (3.2.3) also implies the equi-asymptotic stability of the trivial solution of (3.2.1).

Proof. As in the proof of Theorem 3.4.6, it is easy to see that δ_1 and T depend on t_0, and consequently (S_5) holds.

THEOREM 3.4.9. Assume that there exists a function $V(t, x)$ enjoying the following properties:

(i) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is positive definite, decrescent, and locally Lipschitzian in x.

(ii) $D^+V(t, x) \leqslant -\phi(\| x \|)$, $(t, x) \in J \times S_\rho$, where $\phi \in \mathscr{K}$.

Then, the trivial solution of (3.2.1) is uniformly asymptotically stable.

Proof. Let $0 < \epsilon < \rho$, $t_0 \in J$ be given. Since $V(t, x)$ is positive definite and decrescent, there exist functions $a, b \in \mathscr{K}$ such that

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times S_\rho. \tag{3.4.15}$$

By Corollary 3.3.3, it follows that (S_2) holds. Hence, condition (ii) and Theorem 3.1.3 with $g(t, u) \equiv 0$ give the inequality

$$V(t, x(t)) + \int_{t_0}^{t} \phi(\| x(s)\|)\, ds \leqslant V(t_0, x_0) \tag{3.4.16}$$

for $t \geqslant t_0$. Designate by δ_0 the number $\delta(\rho)$, and choose $T(\epsilon) = a(\delta_0)/\phi(\delta(\epsilon))$, where $\delta(\epsilon)$ corresponds to ϵ in (S_2). Suppose that $\| x_0 \| \leqslant \delta_0$ and that we would have $\| x(t, t_0, x_0)\| \geqslant \delta(\epsilon)$ for $t_0 \leqslant t \leqslant t_0 + T(\epsilon)$. Then, for $t \in [t_0, t_0 + T(\epsilon)]$, we get, from (3.4.16),

$$\begin{aligned} V(t, x(t)) &\leqslant V(t_0, x_0) - \phi(\delta(\epsilon))(t - t_0) \\ &\leqslant a(\delta_0) - \phi(\delta(\epsilon))(t - t_0), \end{aligned}$$

which, for $t = t_0 + T(\epsilon)$, reduces to

$$\begin{aligned} 0 < b(\delta(\epsilon)) &\leqslant V(t_0 + T(\epsilon), x(t_0 + T(\epsilon))) \leqslant a(\delta_0) - \phi(\delta(\epsilon))T(\epsilon) \\ &= 0. \end{aligned}$$

This contradiction proves that there exists a $t_1 \in [t_0, t_0 + T(\epsilon)]$ such that $\| x(t_1, t_0, x_0)\| < \delta(\epsilon)$. Thus, in any case, we have

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0 + T(\epsilon),$$

whenever $\| x_0 \| \leqslant \delta_0$, proving the uniform asymptotic stability of the solution $x = 0$ of (3.2.1).

Notice that condition (ii), together with (3.4.15), yields that

$$D^+V(t, x) \leqslant -\phi[a^{-1}(V(t, x))] \equiv -\phi^*(V(t, x)),$$

where $\phi^* \in \mathscr{K}$, and therefore the conclusion of Theorem 3.4.9 follows right away from Corollary 3.4.2. However, the proof given previously is of interest in other situations.

THEOREM 3.4.10. Assume that there exist functions $V(t, x)$, $g(t, u)$ satisfying the following properties:

(i) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is positive definite, decrescent, and locally Lipschitzian in x.

(ii) $g \in C[J \times R_+, R]$, and, for every pair of numbers α, β such that $0 < \alpha \leqslant \beta < \rho$, there exist $\theta = \theta(\alpha, \beta) \geqslant 0$, $k = k(\alpha, \beta) > 0$ satisfying

$$g(t, u) \leqslant -k, \qquad \alpha \leqslant u \leqslant \beta, \qquad t \geqslant \theta.$$

(iii) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times S_\rho$.

(iv) For any function $\lambda \in C[J, R_+]$,

$$\|f(t, x)\| \leqslant \lambda(t)\| x \|, \qquad (t, x) \in J \times S_\rho. \tag{3.4.17}$$

Then, the trivial solution of (3.2.1) is uniformly asymptotically stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.2.1). Defining $m(t) = \| x(t)\|$, we obtain, from (3.4.17),

$$\begin{aligned} m'_+(t) &\leqslant \| x'(t)\| \\ &\leqslant \|f(t, x(t))\| \leqslant \lambda(t)m(t). \end{aligned}$$

By Theorem 1.4.1, it follows that

$$\| x(t)\| \leqslant \| x_0 \| \exp\left[\int_{t_0}^{t} \lambda(s)\, ds\right],$$

as far as $\| x(t)\| < \rho$. Consider the interval $[t_0, \tau]$ for some $\tau \geqslant t_0$ for which $\| x(t)\| < \rho$. Then, since

$$\int_{t_0}^{\tau} \lambda(s)\, ds \leqslant \int_{0}^{\tau} \lambda(s)\, ds,$$

we obtain

$$\| x(t)\| \leqslant \| x_0 \| e^{N\tau}, \qquad t \in [t_0, \tau], \tag{3.4.18}$$

where $N = N(\tau) = \sup_{0 \leqslant t \leqslant \tau} \lambda(t)$. Since $V(t, x)$ is positive definite and descrescent, (3.4.15) holds. Let $0 < \epsilon < \rho$, $t_0 \in J$ be given. Choose $\delta_1 = \delta_1(\epsilon)$ such that

$$a(\delta_1) < b(\epsilon).$$

It is clear that $\delta_1 < \epsilon$. Let $\theta = \theta[a(\delta_1), b(\epsilon)]$, $k = k[a(\delta_1), b(\epsilon)]$, and $\delta(\epsilon) = \delta_1 e^{-N\theta}$.

We choose $\| x_0 \| < \delta$ so that (3.4.18) assures that $\| x(t)\| < \delta_1$, $t_0 \leqslant t \leqslant \theta$, which implies that

$$V(t, x(t)) \leqslant a(\| x(t)\|) < a(\delta_1), \qquad t_0 \leqslant t \leqslant \theta.$$

We now claim that

$$V(t, x(t)) < b(\epsilon), \qquad t \geqslant \theta.$$

If this were not true, there exist t_1, t_2 such that $t_2 > t_1 \geqslant \theta$, satisfying

$$V(t_1, x(t_1)) = a(\delta_1),$$
$$V(t_2, x(t_2)) = b(\epsilon),$$

and

$$a(\delta_1) \leqslant V(t, x(t)) \leqslant b(\epsilon), \qquad t \in [t_1, t_2]. \tag{3.4.19}$$

Hence, at $t = t_1$, there results

$$D^+ V(t_1, x(t_1)) \geqslant 0. \tag{3.4.20}$$

On the other hand, as $t_1 \geqslant \theta$ and (3.4.19) holds, we obtain, from conditions (ii) and (iii) and the fact that $V(t, x)$ is locally Lipschitzian in x, the inequality

$$\begin{aligned} D^+ V(t_1, x(t_1)) &\leqslant g(t_1, V(t_1, x(t_1))) \\ &\leqslant -k < 0, \end{aligned}$$

which contradicts (3.4.20), thus proving that $V(t, x(t)) < b(\epsilon)$ for $t \geqslant \theta$. It therefore follows that, if $\| x_0 \| < \delta$, $V(t, x(t)) < b(\epsilon)$, $t \geqslant t_0$, and

consequently, in view of (3.4.15), the uniform stability of the trivial solution of (3.2.1) is proved.

Let us denote by δ_0 the number $\delta(\rho)$ obtained by setting $\epsilon = \rho$, and let $0 < \epsilon < \rho$. Let $\delta = \delta(\epsilon)$ be the same function as before. Assume $\| x_0 \| < \delta_0$. Choose

$$T = T(\epsilon) = \frac{a(\delta_0) + c_1\theta_1 + k_1\theta_1}{k_1},$$

where

$$c_1 = \max_{\substack{0 \leqslant t \leqslant \theta_1 \\ b(\delta) \leqslant u \leqslant a(\rho)}} | g(t, u)|,$$

$$\theta_1 = \theta_1[b(\delta), a(\rho)],$$

$$k_1 = k_1[b(\delta), a(\rho)].$$

To prove uniform asymptotic stability, it is enough to show that there exists a $t_1 \in [t_0, t_0 + T]$ satisfying $\| x(t_1)\| < \delta(\epsilon)$; since it would then follow that $\| x(t)\| < \epsilon$, $t \geqslant t_0 + T$ in any case. Suppose there is no such t_1; then,

$$\delta(\epsilon) \leqslant \| x(t)\| \leqslant \rho, \qquad t \in [t_0, t_0 + T],$$

which, because of (3.4.15), shows that

$$b(\delta) \leqslant V(t, x(t)) \leqslant a(\rho), \qquad t \in [t_0, t_0 + T].$$

Using assumptions (i) and (iii), we get

$$\begin{aligned} b(\delta) \leqslant V(t_0 + T, x(t_0 + T)) &\leqslant V(t_0, x_0) \\ &\quad + \int_0^{\theta_1} | g(s, V(s, x(s)))| \, ds + \int_{\theta_1}^{t_0+T} g(s, V(s, x(s))) \, ds \\ &\leqslant a(\delta_0) + c_1\theta_1 - k_1[t_0 + T - \theta_1] \\ &< a(\delta_0) + c_1\theta_1 + k_1\theta_1 - k_1 T = 0. \end{aligned}$$

This absurdity proves that $\| x(t)\| < \delta(\epsilon) < \epsilon$, $t \geqslant t_0 + T$, whenever $\| x_0 \| < \delta_0$, and the theorem is completely proved.

We have seen in Theorem 3.3.5 that the uniform stability of the trivial solution of (3.2.1) can be formulated by means of monotone functions. We may likewise state the following theorem with respect to uniform asymptotic stability.

THEOREM 3.4.11. The trivial solution of (3.2.1) is uniformly asymptotically stable if and only if there exist functions $a \in \mathscr{K}$, $\sigma \in \mathscr{L}$ such that

$$\| x(t, t_0, x_0)\| \leqslant a(\| x_0 \|)\sigma(t - t_0), \qquad t \geqslant t_0, \tag{3.4.21}$$

for $\| x_0 \| < \rho$.

Proof. If (3.4.21) is satisfied, it is easy to verify that (S_6) holds, and hence sufficiency of the condition is evident.

Assume now that the trivial solution of (3.2.1) is uniformly asymptotically stable, so that (S_6) holds. Let $\{\epsilon_n\}$ be a positive, monotonic sequence, converging to zero as $n \to \infty$. Let

$$T_1(\epsilon) = \inf T(\epsilon_{n+1}) \qquad \text{for} \quad \epsilon_{n+1} \leqslant \epsilon \leqslant \epsilon_n .$$

Then,

$$\| x_0 \| < \delta_0, \qquad t \geqslant t_0 + T_1(\epsilon)$$

assures that

$$\| x(t, t_0, x_0)\| < \epsilon_{n+1} \leqslant \epsilon.$$

Define $T^*(\epsilon)$ linear in $[\epsilon_{n+1}, \epsilon_n]$, $T^*(\epsilon_{n+1}) = T_1(\epsilon_{n+2})$, and $T^*(\epsilon_n) = T_1(\epsilon_{n+1})$. Note that

$$T_1(\epsilon_{n+1}) \geqslant T_1(\epsilon_n), \qquad \lim_{n\to\infty} T_1(\epsilon_n) = \infty.$$

The equality $T_1(\epsilon_{n+1}) = T_1(\epsilon_n)$ may occur on finite parts of the sequence which can be eliminated. The function $T^*(\epsilon)$ is continuous, monotone decreasing, and $\lim_{\epsilon\to 0} T^*(\epsilon) = \infty$. Moreover, $t \geqslant t_0 + T^*(\epsilon)$ implies

$$t \geqslant t_0 + T^*(\epsilon_n) = t_0 + T_1(\epsilon_{n+1}).$$

Hence, if $\epsilon_{n+1} \leqslant \epsilon \leqslant \epsilon_n$, we have

$$\| x(t, t_0, x_0)\| < \epsilon_{n+1} \leqslant \epsilon, \qquad t \geqslant t_0 + T^*(\epsilon).$$

This shows that the function $T(\epsilon)$ occurring in (S_4) can be chosen continuous and monotonic. Let $\eta(T)$ be the inverse function of $T(\epsilon)$. Then, it follows that

$$\| x(t, t_0, x_0)\| \leqslant \eta(t - t_0), \qquad t \geqslant t_0,$$

if $\| x_0 \| < \delta_0$. Also, by Theorem 3.3.5, we obtain that

$$\| x(t, t_0, x_0)\| \leqslant \epsilon(\| x_0 \|), \qquad t \geqslant t_0,$$

where $\epsilon(\delta)$ is the inverse function of $\delta(\epsilon)$. Thus,

$$\| x(t, t_0, x_0)\|^2 \leqslant \epsilon(\| x_0 \|)\eta(t - t_0), \qquad t \geqslant t_0,$$

whenever $\| x_0 \| < \delta_0$, which proves that uniform asymptotic stability is equivalent to (3.4.21).

3.5. Stability of perturbed systems

We shall consider the perturbed system

$$x' = f(t, x) + R(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{3.5.1}$$

where $f, R \in C[J \times S_\rho, R^n]$.

THEOREM 3.5.1. Let there exist a function $V(t, x)$ satisfying the following conditions:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ possesses continuous partial derivatives with respect to t and the components of x, and

$$\left\| \frac{\partial V(t, x)}{\partial x} \right\| \leqslant L\| x \|^{\alpha-1}, \qquad \alpha \geqslant 1.$$

(ii)

$$V'(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} \cdot f(t, x) \leqslant -C\| x \|^{\alpha+m-1}, m \geqslant 1,$$

for $(t, x) \in J \times S_\rho$.

(iii) $A \| x \|^\alpha \leqslant V(t, x) \leqslant B \| x \|^\alpha$, A, B being positive constants.

(iv) $w \in C[J \times R_+, R_+]$, $w(t, 0) \equiv 0$, $w(t, u)$ is monotone nondecreasing in u for each t, and

$$\| R(t, x)\| \leqslant w(t, \| x \|^m).$$

Then, the stability properties of the trivial solution of (3.2.3), with

$$g(t, u) = -\frac{C}{A\alpha}\left(\frac{A}{B}\right)^{(\alpha+m-1)/\alpha} u^m + \frac{L}{A\alpha} w(t, u^m), \tag{3.5.2}$$

imply the corresponding stability properties of the solution $x = 0$ of (3.5.1).

Proof. Consider the function

$$U(t, x) = \left[\frac{V(t, x)}{A}\right]^{1/\alpha}.$$

In view of the assumptions of the theorem, if we define

$$U'(t, x) = \frac{\partial U(t, x)}{\partial t} + \frac{\partial U(t, x)}{\partial x} \cdot [f(t, x) + R(t, x)],$$

we obtain, after some computations, that

$$\| x \| \leqslant U(t, x) \leqslant B^{1/\alpha} \| x \|, \qquad (t, x) \in J \times S_\rho,$$
$$U'(t, x) \leqslant g(t, U(t, x)), \qquad (t, x) \in J \times S_\rho,$$

where $g(t, u)$ is the same function given by (3.5.2). The conclusion of the theorem is then a direct conseqeunce of Theorems 3.3.3, 3.3.4, 3.4.3, and 3.4.4.

COROLLARY 3.5.1. Let the assumptions of Theorem 3.5.1 hold with $\alpha = 2$, $m = 1$, and $w(t, u) = \lambda(t)u$, where $\lambda(t) \geqslant 0$ is continuous on J such that

$$\limsup_{t\to\infty} \left[\frac{1}{t - t_0} \int_{t_0}^{t} \lambda(s)\, ds\right] < CA/BL.$$

Then, the solution $x = 0$ of (3.5.1) is asymptotically stable.

THEOREM 3.5.2. Let there exist functions $V(t, x)$, $g_1(t, u)$, and $w(t, u)$ fulfilling the following conditions:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is Lipschitzian in x for a function $k(t) \geqslant 0$ continuous on J, and

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|), \qquad t, x) \in J \times S_\rho, \tag{3.5.3}$$

where $a, b \in \mathscr{K}$.

(ii) $g_1 \in C[J \times R_+, R]$, $g_1(t, 0) \equiv 0$, and

$$D^+V(t, x)_{(3.2.1)} \leqslant g_1(t, V(t, x)), \qquad (t, x) \in J \times S_\rho,$$

(iii) $w \in C[J \times R_+, R_+]$, $w(t, 0) \equiv 0$, $w(t, u)$ is nondecreasing in u for each t, and

$$\| R(t, x)\| \leqslant w(t, \| x \|), \qquad (t, x) \in J \times S_\rho.$$

Then, the stability properties of the solution $u = 0$ of (3.2.3) with

$$g(t, u) = g_1(t, u) + k(t)w(t, b^{-1}(u)), \tag{3.5.4}$$

where $b^{-1}(u)$ is the inverse function of $b(u)$, imply the same kind of stability properties of the trivial solution of (3.5.1).

Proof. Let us define the function $D^+V(t, x)$ with respect to the perturbed differential system (3.5.1) as follows:

$$D^+V(t, x)_{(3.5.1)} = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + h[f(t, x) + R(t, x)]) - V(t, x)].$$

Then, since $V(t, x)$ is Lipschitzian in x for a function $k(t)$, we have, for $(t, x) \in J \times S_\rho$,

$$\begin{aligned} D^+V(t, x)_{(3.5.1)} &\leqslant D^+V(t, x)_{(3.2.1)} + k(t)\| R(t, x)\| \\ &\leqslant g_1(t, V(t, x)) + k(t)\, w(t, \| x \|), \end{aligned}$$

using assumptions (ii) and (iii). This, together with (3.5.3) and the monotonicity of $w(t, u)$ in u, leads to the differential inequality

$$D^+V(t, x)_{(3.5.1)} \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times S_\rho,$$

where $g(t, u)$ is given by (3.5.4). Now, it only remains to apply Theorems 3.3.3, 3.3.4, 3.4.3, and 3.4.4 to get the desired result.

COROLLARY 3.5.2. The functions $b(u) = u, g_1(t, u) = -\alpha u,\ \alpha > 0$, and $w(t, u) = \lambda(t)u, \lambda(t) \geqslant 0$ being continuous on J and satisfying

$$\limsup_{t \to \infty} \left[\frac{1}{t - t_0} \int_{t_0}^{t} \lambda(s)\, ds\right] < \alpha/k,$$

are admissible in Theorem 3.5.2, to guarantee the uniform asymptotic stability of the solution $x = 0$ of (3.5.1) provided k is the Lipschitz constant for $V(t, x)$.

COROLLARY 3.5.3. The functions $w(t, u) = \lambda(t)\,\phi(u),\ \ g_1(t, u) \equiv 0$, where $\lambda \in C[J, R_+], \phi \in \mathscr{K}$, are admissible in Theorem 3.5.2 to yield the uniform stability of the trivial solution of (3.5.1), provided that k is the Lipschitz constant for $V(t, x)$ and $\int_0^\infty \lambda(s)\, ds < \infty$.

COROLLARY 3.5.4. The functions $w(t, u) = \lambda(t)\,\phi(u), g_1(t, u) = -C(u)$, where $\lambda \in C[J, R_+], \phi, C \in \mathscr{K}$, are admissible in Theorem 3.5.2 to assure that the trivial solution of (3.5.1) is uniformly asymptotically stable, provided that k is the Lipschitz constant for $V(t, x)$, $k\phi(b^{-1}(u)) \leqslant \alpha C(u)$, for some $\alpha > 0$, and $\lim_{t \to \infty} [-t + \alpha \int_{t_0}^{t_0+t} \lambda(s)\, ds] = -\infty$ for all $t_0 \geqslant 0$.

3.6. Converse theorems

This section will be devoted to a variety of results concerning the construction of Lyapunov functions. Let us first define the notion of generalized exponential asymptotic stability.

DEFINITION 3.6.1. The trivial solution of (3.2.1) is said to be (S_{11}) *generalized exponentially asymptotically stable* if

$$\| x(t, t_0, x_0)\| \leqslant K(t_0)\| x_0 \| \exp[p(t_0) - p(t)], \qquad t \geqslant t_0, \tag{3.6.1}$$

where $K(t) > 0$ is continuous for $t \in J$, $p \in \mathscr{K}$ for $t \in J$, and $p(t) \to \infty$ as $t \to \infty$.

In particular, if $K(t) = K > 0$, $p(t) = \alpha t$, $\alpha > 0$. We have the *exponential asymptotic stability* of the trivial solution of (3.2.1)

THEOREM 3.6.1. Assume that the solution $x = 0$ of (3.2.1) is generalized exponentially asymptotically stable and that $f(t, x)$ is linear in x. Suppose further that $p'(t)$ exists and is continuous on J. Then there exists a function $V(t, x)$ satisfying the following properties:

(i) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is Lipschitzian in x for the function $K(t)$.

(ii) $\| x \| \leqslant V(t, x) \leqslant K(t) \| x \|$, $(t, x) \in J \times S_\rho$.

(iii) $D^+V(t, x) \leqslant -p'(t) V(t, x)$, $(t, x) \in J \times S_\rho$.

Proof. Define

$$V(t, x) = \sup_{\sigma \geqslant 0} \| x(t + \sigma, t, x)\| \exp[p(t + \sigma) - p(t)]. \tag{3.6.2}$$

Then, from (3.6.1), it follows that (ii) is satisfied.

Let $x, y \in S_\rho$. Then,

$$\begin{aligned} |V(t, x) - V(t, y)| &= |\sup_{\sigma \geqslant 0} \| x(t + \sigma, t, x)\| \exp[p(t + \sigma) - p(t)] \\ &\qquad - \sup_{\sigma \geqslant 0} \| x(t + \sigma, t, y)\| \exp[p(t + \sigma) - p(t)]| \\ &\leqslant \sup_{\sigma \geqslant 0} [\| x(t + \sigma, t, x) - x(t + \sigma, t, y)\| \exp(p(t + \sigma) - p(t))] \\ &= \sup_{\sigma \geqslant 0} \| x(t + \sigma, t, x - y)\| \exp[p(t + \sigma) - p(t)] \\ &\leqslant K(t)\| x - y \|. \end{aligned}$$

To arrive at this estimate, we have used the facts that $f(t, x)$ is linear in x and that the solutions $x(t, t_0, x_0)$ obey the inequality (3.6.1).

We shall now prove that $V(t, x)$ is continuous. Let $\delta \geqslant 0$; then

$$\begin{aligned} | V(t+\delta, x^*) - V(t, x) \leqslant | V(t+\delta, x^*) - V(t+\delta, x)| \\ + | V(t+\delta, x) - V(t+\delta, x(t+\delta, x(t+\delta, t, x)))| \\ + | V(t+\delta, x(t+\delta, t, x)) - V(t, x)|. \end{aligned}$$

Since $V(t, x)$ is Lipschitzian in x and $x(t+\delta, t, x)$ is continuous in δ, the first two terms on the right-hand side of the preceding inequality are small when $\| x - x^* \|$ and δ are small. Let us consider the third term. Observe that

$$x(t+\delta+\sigma, t+\delta, x(t+\delta, t, x)) = x(t+\delta+\sigma, t, x).$$

Hence we have

$$\begin{aligned} &| V(t+\delta, x(t+\delta, t, x)) - V(t, x)| \\ &\quad = | \sup_{\sigma \geqslant 0} \| x(t+\delta+\sigma, t+\delta, x(t+\delta, t, x))\| \exp[p(t+\sigma) - p(t)] \\ &\quad\quad - \sup_{\sigma \geqslant 0} \| x(t+\sigma, t, x)\| \exp[p(t+\sigma) - p(t)]| \\ &\quad = | \sup_{\sigma \geqslant \delta} \| x(t+\sigma, t, x)\| \exp[p(t+\sigma) - p(t+\delta)] \\ &\quad\quad - \sup_{\sigma \geqslant 0} \| x(t+\sigma, t, x) \exp[p(t+\sigma) - p(t)]|. \end{aligned}$$

Setting

$$a(\delta) = \sup_{\sigma \geqslant \delta} \| x(t+\sigma, t, x)\|[p(t+\sigma) - p(t+\delta)],$$

we notice that $a(\delta)$ is nondecreasing and tends to $a(0)$ as $\delta \to 0$, since $\| x(t+\sigma, t, x)\| \exp[p(t+\sigma) - p(t)]$ is a bounded continuous function for all $\sigma \geqslant 0$. Thus,

$$| V(t+\delta, x(t+\delta, t, x)) - V(t, x)| = | a(\delta) - a(0)|$$

implies that the third term tends to zero as $\delta \to 0$. Thus we have verified the continuity of $V(t, x)$.

Furthermore, using the uniqueness of solutions and the definition (3.6.2),

$$\begin{aligned}
D^+V(t, x(t)) &= \limsup_{h\to 0^+} \frac{1}{h} [V(t + h, x(t + h, t, x)) - V(t, x)] \\
&= \limsup_{h\to 0^+} \frac{1}{h} [\sup_{\sigma\geqslant 0} \| x(t + h + \sigma, t + h, x(t + h, t, x))\| \\
&\quad \times \exp[p(t + h + \sigma) - p(t + h)] - \sup_{\sigma\geqslant 0} \| x(t + \sigma, t, x)\| \\
&\quad \times \exp[p(t + \sigma) - p(t)]] \\
&= \limsup_{h\to 0^+} \frac{1}{h} \sup_{\sigma\geqslant h} [\| x(t + \sigma, t, x)\| \exp(p(t + \sigma) - p(t + h)) \\
&\quad - \sup_{\sigma\geqslant 0} \| x(t + \sigma, t, x)\| \exp(p(t + \sigma) - p(t))] \\
&\leqslant \limsup_{h\to 0^+} \frac{1}{h} [\sup_{\sigma\geqslant 0} \| x(t + \sigma, t, x)\| \exp(p(t + \sigma) - p(t)) \\
&\quad \times \{\exp(p(t) - p(t + h)) - 1\}] \\
&\leqslant -p'(t)V(t, x).
\end{aligned}$$

Since, for small $h > 0$,

$$\begin{aligned}
V(t + h, x + hf(t, x)) - V(t, x) \leqslant K(t) \| x(t + h, t, x) - x - hf(t, x)\| \\
+ V(t + h, x(t + h, t, x)) - V(t, x),
\end{aligned}$$

it easily follows that

$$D^+V(t, x) \leqslant -p'(t)V(t, x),$$

proving (iii). The theorem is completely proved.

A similar result is true even when $f(t, x)$ is nonlinear in x provided it is assumed to satisfy a Lipschitz condition in x for a constant L. The next theorem substantiates this remark.

THEOREM 3.6.2. Let the trivial solution of (3.2.1) be generalized exponentially asymptotically stable and $f(t, x)$ satisfy a Lipschitz condition in x for a constant $L = L(\rho) > 0$. Assume that $p'(t)$ exists and is continuous for $t \in J$. Suppose further that $K(t)$ is bounded and, for some q, $0 < q < 1$, there exists a number $T > 0$ such that

$$K(t) \exp[-q(p(t + T) - p(t))] \leqslant 1, \qquad t \in J. \tag{3.6.3}$$

Then, there exists a function $V(t, x)$ fulfilling the following conditions:

(i) $V \in C[J \times S_{\rho_0}, R_+]$, $0 < \rho_0 \leqslant \rho$, and $V(t, x)$ obeys

$$| V(t, x) - V(t, y)| \leqslant e^{LT} \sup_{0 \leqslant \sigma \leqslant T} \exp[(1 - q)\{p(t + \sigma) - p(t)\}] \| x - y \|.$$

(ii) $\| x \| \leqslant V(t, x) \leqslant K(t) \| x \|$, $(t, x) \in J \times S_{\rho_0}$.

(iii) $D^+V(t, x) \leqslant -(1 - q) p'(t) V(t, x)$, $(t, x) \in J \times S_{\rho_0}$.

Proof. Let q, T be given so that the relation (3.6.3) holds. Define

$$V(t, x) = \sup_{\sigma \geqslant 0} \| x(t + \sigma, t, x)\| \exp[(1 - q)\{p(t + \sigma) - p(t)\}]. \quad (3.6.4)$$

Since $K(t)$ is assumed to be bounded and $p(t) \in \mathscr{K}$, the function $V(t, x)$ is defined for $(t, x) \in J \times S_{\rho_0}$, where $\rho_0 = \rho/M$, $M = \sup_{t \in J} K(t)$. To prove that $V(t, x)$ satisfies the Lipschitz condition, we first observe that

$$\begin{aligned} &\| x(t + \sigma, t, x)\| \exp[(1 - q)\{p(t + \sigma) - p(t)\}] \\ &\quad \leqslant K(t) \exp[-q(p(t + \sigma) - p(t))] \| x \|, \end{aligned}$$

and, since (3.6.3) holds, we have

$$V(t, x) = \sup_{0 \leqslant \sigma \leqslant T} \| x(t + \sigma, t, x)\| \exp[(1 - q)\{p(t + \sigma) - p(t)\}].$$

Accordingly, by Corollary 2.7.1, we get

$$\begin{aligned} | V(t, x) - V(t, y)| &\leqslant \sup_{0 \leqslant \sigma \leqslant T} [\| x(t + \sigma, t, x) - x(t + \sigma, t, y)\| \\ &\qquad \times \exp\{(1 - q)(p(t + \sigma) - p(t))\}] \\ &\leqslant e^{LT} \sup_{0 \leqslant \sigma \leqslant T} [\exp(1 - q)\{p(t + \sigma) - p(t)\}] \| x - y \|. \end{aligned}$$

The relations (ii) and (iii) can be verified as in the proof of Theorem 3.6.1. The proof is complete.

COROLLARY 3.6.1. If the trivial solution $x = 0$ of (3.2.1) is exponentially asymptotically stable, that is, $p(t) \equiv \alpha t$, $\alpha > 0$, $K(t) \equiv K > 0$, then $T = \log K/q\alpha$, and $M = K^{L+(-q)\alpha/q\alpha}$, where M is the Lipschitz constant for $V(t, x)$, are admissible in Theorem 3.6.2.

It is possible to prove the previous theorems, under milder assumptions, in a different way.

THEOREM 3.6.3. Assume that

(i) $f \in C[J \times S_\rho, R^n], f(t, 0) \equiv 0$, and $f(t, x)$ satisfies

$$\|f(t, x) - f(t, y)\| \leqslant L(t) \| x - y \|, \qquad (t, x), (t, y) \in J \times S_\rho, \quad (3.6.5)$$

$L(t) \geqslant 0$ being continuous on J;

(ii) There exists a $p \in \mathscr{K}$ for $t \in J, p(t) \to \infty$ as $t \to \infty$, $p'(t)$ exists, and

$$\| x(t, 0, x_0)\| \leqslant K \| x_0 \| \exp[-p(t)], \qquad t \geqslant t_0, K > 0, \qquad (3.6.6)$$

where $x(t, 0, x_0)$ is the solution of (3.2.1) through $(0, x_0)$.

Then, there exists a function $V(t, x)$ enjoying the following properties:

(1) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is Lipschitzian in x for a continuous function $K(t) > 0$.

(2) $\| x \| \leqslant V(t, x) \leqslant K(t) \| x \|, (t, x) \in J \times S_\rho$.

(3) $D^+V(t, x) \leqslant -p'(t) V(t, x), (t, x) \in J \times S_\rho$.

Proof. Let us denote $x = x(t, 0, x_0)$ so that $x_0 = x(0, t, x)$, because of the uniqueness of solutions of (3.2.1), which is assured by condition (i). We now define

$$V(t, x) = Ke^{-p(t)} \| x(0, t, x)\|. \qquad (3.6.7)$$

It is then evident that $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$, and, because of (3.6.6), we have

$$\| x \| \leqslant V(t, x).$$

Since the solutions of (3.2.1) are unique, it follows that

$$\begin{aligned} V(t + h, x(t + h, t, x)) &= Ke^{-p(t+h)} \| x(0, t + h, x(t + h, t, x))\| \\ &= Ke^{-p(t+h)} \| x(0, t, x)\|, \end{aligned}$$

and hence we obtain

$$\begin{aligned} D^+V(t, x(t)) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x(t + h, t, x)) - V(t, x)] \\ &= -p'(t)V(t, x). \end{aligned} \qquad (3.6.8)$$

If $x(t, 0, x_0)$, $x(t, 0, y_0)$ are the two solutions of (3.2.1) through $(0, x_0)$, $(0, y_0)$, respectively, the condition (3.6.5) yields

$$\| x_0 - y_0 \| \exp\left[-\int_0^t L(s)\, ds\right] \leqslant \| x(t, 0, x_0) - x(t, 0, y_0)\|, \qquad t \geqslant 0,$$

by virtue of Corollary 2.7.1. Letting $x = x(t, 0, x_0)$, $y = x(t, 0, y_0)$, we get, from the preceding estimate, the inequality

$$\| x(0, t, x) - x(0, t, y)\| \leqslant \| x - y \| \exp\left[\int_0^t L(s)\, ds\right], \qquad t \geqslant 0.$$

Consequently,

$$\begin{aligned} | V(t, x) - V(t, y) | &= Ke^{-p(t)} \| x(0, t, x) - x(0, t, y)\| \\ &\leqslant K \exp\left[-p(t) + \int_0^t L(s)\, ds\right] \| x - y \| \\ &= K(t) \| x - y \|, \end{aligned}$$

if we define $K(t) = K \exp[-p(t) + \int_0^t L(s)\, ds]$. This proves that $V(t, x)$ satisfies a Lipschitz condition in x for a function $K(t) > 0$. The upper estimate in (2) follows from the Lipschitz condition by setting $y = 0$ and observing that $V(t, 0) \equiv 0$. As in the proof of Theorem 3.6.1, one can deduce (3) from (3.6.8). The theorem is proved.

COROLLARY 3.6.2. If in Theorem 3.6.3, the functions $L(t)$ and $p(t)$ are such that $\int_0^t L(s)\, ds \leqslant p(t)$, and $V(t, x)$ satisfies the Lipschitz condition in x for a constant $K > 0$.

We have already seen that the concepts of stability and asymptotic stability can be defined by means of simple inequalities involving certain monotone functions. We give below some converse theorems in terms of differential inequalities. As will be seen, the approach depends upon the differentiable properties of solutions with respect to the initial values and yields, in a unified way, a method of constructing Lyapunov functions.

THEOREM 3.6.4. Suppose that

(i) the function $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $\partial f(t, x)/\partial x$ exists and is continuous for $(t, x) \in J \times S_\rho$;

(ii) there exist functions $\beta_1, \beta_2 \in \mathscr{K}$ such that

$$\beta_1(\| x_0 \|) \leqslant \| x(t, 0, x_0)\| \leqslant \beta_2(\| x_0 \|), \qquad t \in J, \tag{3.6.9}$$

where $x(t, 0, x_0)$ is the solution of (3.2.1) through $(0, x_0)$;

(iii) the function $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and $\partial g(t, u)/\partial u$ exists and is continuous for $(t, u) \in J \times R_+$;

(iv) the solution $u(t, 0, u_0)$ of (3.2.3) fulfills the estimate

$$\gamma_1(u_0) \leqslant u(t, 0, u_0) \leqslant \gamma_2(u_0), \qquad t \in J, \tag{3.6.10}$$

where $\gamma_1, \gamma_2 \in \mathscr{K}$.

Then, there exists a function $V(t, x)$ with the following properties:

(1) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ possesses continuous partial derivatives with respect to t and the components of x.

(2) $b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|)$, $(t, x) \in J \times S_\rho$, $a, b \in \mathscr{K}$.

(3)
$$\begin{aligned} V'(t, x) &= \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} \cdot f(t, x) \\ &= g(t, V(t, x)), (t, x) \in J \times S_\rho . \end{aligned}$$

Proof. In view of hypothesis (i), the existence and uniqueness of solutions of (3.2.1), as well as their continuous dependence on the initial values, is assured. Also, the solutions $x(t, t_0, x_0)$, $(t_0, x_0) \in J \times S_\rho$ are differentiable functions with respect to the initial values, and $x = 0$ is the trivial solution. Furthermore, on the basis of Theorem 2.5.3, we have

$$\frac{\partial x(t_0, t_0, x_0)}{\partial t_0} = -f(t_0, x_0)$$

and

$$\frac{\partial x(t, t_0, x_0)}{\partial t_0} = -\Phi(t, t_0, x_0) f(t_0, x_0), \qquad t \geqslant t_0, \tag{3.6.11}$$

where $\Phi(t, t_0, x_0) = \partial x(t, t_0, x_0)/\partial x_0$ is the fundamental matrix solution of the variational system

$$y' = \left[\frac{\partial f(t, x(t, t_0, x_0))}{\partial x}\right] y,$$

such that $\Phi(t_0, t_0, x_0)$ is the unit matrix. Similar conclusions hold for the solutions of (3.2.3) as (iii) is satisfied.

Let now $x(t, 0, x_0)$, $u(t, 0, u_0)$ be the solutions of (3.2.1), (3.2.3) through $(0, x_0)$, $(0, u_0)$, satisfying the inequalities (3.6.9) and (3.6.10), respectively. Denote $x(t, 0, x_0)$ by x so that $x_0 = x(0, t, x)$. This is clear by virtue of the uniqueness of solutions of (3.2.1).

Choose any continuous function $\mu(x)$ possessing continuous partial derivatives $\partial\mu(x)/\partial x$ for $x \in S_\rho$ such that

$$\alpha_1(\| x \|) \leqslant \mu(x) \leqslant \alpha_2(\| x \|), \qquad \alpha_1, \alpha_2 \in \mathscr{K}. \tag{3.6.12}$$

Define the function

$$V(t, x) = u[t, 0, \mu(x(0, t, x))].$$

Because of the continuity of $x(0, t, x)$, $\mu(x)$, and $u(t, 0, u_0)$ with respect to their arguments and the fact that $u = 0$ is the trivial solution of (3.2.3), it follows that $V \in C[J \times S_\rho, R_+]$. Since the functions $x(0, t, x)$, $u(t, 0, u_0)$ are differentiable with respect to the initial values and $\partial\mu(x)/\partial x$ is assumed to exist, we see that

$$\begin{aligned} V'(t, x) &= \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} \cdot f(t, x) \\ &= u'[t, 0, \mu(x(0, t, x))] \\ &\quad + \frac{\partial u}{\partial u_0} \frac{\partial \mu}{\partial x} \left[\frac{\partial x(0, t, x)}{\partial t_0} + \frac{\partial x(0, t, x)}{\partial x_0} \cdot f(t, x) \right] \\ &= g[t, u[t, 0, \mu(x(0, t, x))]) \\ &= g(t, V(t, x)), \end{aligned}$$

because, by relation (3.6.11), we have

$$\frac{\partial x(0, t, x)}{\partial t_0} + \frac{\partial x(0, t, x)}{\partial x_0} \cdot f(t, x) = 0.$$

Thus, (1) and (3) hold.

Since $x = x(t, 0, x_0)$ and $x_0 = x(0, t, x)$, it follows from the inequality (3.6.9) that

$$\beta_2^{-1}(\| x \|) \leqslant \| x(0, t, x) \| \leqslant \beta_1^{-1}(\| x \|), \tag{3.6.13}$$

where β_1^{-1}, β_2^{-1} are inverse functions of β_1, β_2 and hence belong to $\mathscr{K}$. Using the inequalities (3.6.10), (3.6.12), and (3.6.13) successively, the definition of $V(t, x)$ gives

$$\begin{aligned} V(t, x) &= u[t, 0, \mu(x(0, t, x))] \\ &\geqslant \gamma_1(\mu(x(0, t, x))) \\ &\geqslant \gamma_1(\alpha_1(\| x(0, t, x) \|)) \\ &\geqslant \gamma_1(\alpha_1(\beta_2^{-1}(\| x \|))) \\ &\equiv b(\| x \|) \end{aligned}$$

and

$$\begin{aligned} V(t, x) &\leqslant \gamma_2(\mu(x(0, t, x))) \\ &\leqslant \gamma_2(\alpha_2(\| x(0, t, x) \|)) \\ &\leqslant \gamma_2(\alpha_2(\beta_1^{-1}(\| x \|))) \\ &\equiv a(\| x \|). \end{aligned}$$

Obviously, the functions $a, b \in \mathscr{K}$, since the functions γ_1, α_1, β_2^{-1}, γ_2, α_2, and β_1^{-1} all belong to class $\mathscr{K}$.
This proves (2), and the proof is complete.

COROLLARY 3.6.3. Assume that $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $\partial f(t, x)/\partial x$ exists and is continuous for $(t, x) \in J \times S_\rho$. Suppose also that, for any solution $x(t, 0, x_0)$ through the point $(0, x_0)$ of (3.2.1),

$$\| x(t, 0, x_0)\| \leqslant \beta_2(\| x_0 \|), \qquad t \geqslant 0, \quad \beta_2 \in \mathscr{K}.$$

Then, there exists a function $V(t, x)$ such that:

(1) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) = 0$, and $V(t, x)$ possesses continuous partial derivatives with respect to t and the components of x;

(2) $b(\| x \|) \leqslant V(t, x)$, $(t, x) \in J \times S_\rho$, $b \in \mathscr{K}$;

(3)

$$\begin{aligned} V'(t, x) &= \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} \cdot f(t, x) \\ &= 0, \quad (t, x) \in J \times S_\rho . \end{aligned}$$

The following variant of Theorem 3.6.4 is of interest in some situations.

THEOREM 3.6.5. Let

(i) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $f(t, x)$ satisfies

$$\|f(t, x) - f(t, y)\| \leqslant L_1(t) \| x - y \|$$

for (t, x), $(t, y) \in J \times S_\rho$, where $L_1(t) \geqslant 0$ is continuous on J;

(ii) there exists a function $\beta_2 \in \mathscr{K}$ such that

$$\| x(t, 0, x_0)\| \leqslant \beta_2(\| x_0 \|), \qquad t \geqslant 0,$$

$x(t, 0, x_0)$ being the solution of (3.2.1);

(iii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and $g(t, u)$ verifies

$$| g(t, u) - g(t, v)| \leqslant L_2(t) \, | u - v |$$

for $t \in J$, $u, v \geqslant 0$, $L_2(t) \geqslant 0$ being continuous on J;

(iv) the solution $u(t, 0, u_0)$ of (3.2.3) fulfill the estimate

$$\gamma_1(u_0) \leqslant u(t, 0, u_0), \qquad t \geqslant 0, \quad \gamma_1 \in \mathscr{K}.$$

Then, there exists a function $V(t, x)$ with the following properties:

(1) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is positive definite and satisfies a Lipschitz condition for a continuous function $K(t) \geqslant 0$.

(2) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times S_\rho$.

Proof. Since the uniqueness of solutions, as well as their continuous dependence on initial values, is guaranteed by (i) on the basis of Corollary 2.5.1, if we let $x = x(t, 0, x_0)$, it follows that $x_0 = x(0, t, x)$ as previously. We define

$$V(t, x) = u(t, 0, \| x(0, t, x)\|),$$

where $u(t, 0, u_0)$ is the solution of (3.2.3) through $(0, u_0)$. Note that, by assumption (iii), $u = 0$ is the trivial solution of (3.2.3) and that the solutions $u(t, t_0, u_0)$ are unique. Furthermore, we infer from uniqueness of solutions of (3.2.1) that

$$\begin{aligned} V(t + h, x(t + h, t, x)) &= u(t + h, 0, \| x(0, t + h, x(t + h, t, x))\|) \\ &= u(t + h, 0, \| x(0, t, x)\|), \end{aligned}$$

and therefore

$$\begin{aligned} D^+V(t, x(t)) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x(t + h, t, x)) - V(t, x)] \\ &= u'(t, 0, \| x(0, t, x)\|) \\ &= g(t, V(t, x)). \end{aligned} \tag{3.6.14}$$

According to Corollary 2.7.1, if we let $x = x(t, 0, x_0)$, $y = x(t, 0, y_0)$, condition (i) implies, as in Theorem 3.6.3, the inequality

$$\| x(0, t, x) - x(0, t, y)\| \leqslant \| x - y \| \exp \left[\int_0^t L_1(s)\, ds\right], \qquad t \geqslant 0.$$

Moreover, assumption (iii) also implies that

$$| u(t, 0, u_0) - u(t, 0, v_0)| \leqslant | u_0 - v_0 | \exp \left[\int_0^t L_2(s)\, ds\right], \qquad t \geqslant 0.$$

Thus, for (t, x), $(t, y) \in J \times S_\rho$, we have

$$\begin{aligned} | V(t, x) - V(t, y)| &= | u(t, 0, \| x(0, t, x)\|) - u(t, 0, \| x(0, t, y)\|)| \\ &\leqslant \| x(0, t, x) - x(0, t, y)\| \exp \left[\int_0^t L_2(s)\, ds\right] \\ &\leqslant \| x - y \| \exp \left[\int_0^t (L_1(s) + L_2(s))\, ds\right] \\ &\equiv K(t) \| x - y \|, \end{aligned}$$

if $K(t) = \exp[\int_0^t (L_1(s) + L_2(s))\, ds]$. This, together with (3.6.14), enables us to deduce, as in Theorem 3.6.1, that

$$D^+V(t, x) \leqslant g(t, V(t, x)),$$

proving (2).

To show that $V(t, x)$ is positive definite, we use the estimates given in (ii) and (iv) and obtain

$$\begin{aligned} V(t, x) &= u(t, 0, \| x(0, t, x)\|) \\ &\geqslant \gamma_1(\| x(0, t, x)\|) \\ &\geqslant \gamma_1(\beta_2^{-1}(\| x \|)) \\ &= b(\| x \|), \end{aligned}$$

where $b \in \mathscr{K}$ since $\gamma_1, \beta_2^{-1} \in \mathscr{K}$.

COROLLARY 3.6.4. The function $g(t, u) \equiv 0$ is admissible in Theorem 3.6.5.

The next theorem deals with the converse problem for asymptotic stability.

THEOREM 3.6.6. Let assumptions (i) and (iv) of Theorem 3.6.4 hold. Suppose that the solution $x(t, 0, x_0)$ of (3.2.1) satisfies

$$\| x(t, 0, x_0)\| \leqslant \beta_2(\| x_0 \|)\sigma(t), \qquad t \geqslant 0, \tag{3.6.15}$$

where $\beta_2 \in \mathscr{K}$, $\sigma \in \mathscr{L}$. Assume that there exist functions $\gamma \in \mathscr{K}$, $\delta \in \mathscr{L}$ such that

$$\gamma'(u) \geqslant k > 0, \qquad \delta(t) \geqslant k_1\sigma(t), \qquad k_1 > 0,$$

and

$$\gamma(u_0)\delta(t) \leqslant u(t, 0, u_0), \qquad t \geqslant 0. \tag{3.6.16}$$

Then, there exists a function $V(t, x)$ satisfying

(1) $V \in C[J \times S_\rho, R_+]$, $\quad V(t, 0) \equiv 0$,
$V(t, x)$ is positive definite and possesses continuous partial derivatives with respect to t and the components of x;

(2) $V'(t, x) = g(t, V(t, x))$, $\quad (t, x) \in J \times S_\rho$.

Proof. Let $x(t, 0, x_0)$, $u(t, 0, u_0)$ be the solutions of (3.2.1), (3.2.3) obeying the estimates (3.6.15), (3.6.16), respectively. Choose any

continuous function $\mu(x)$ having continuous partial derivatives $\partial\mu(x)/\partial x$ for $x \in S_\rho$ such that $\mu(0) = 0$ and

$$\beta_2(\| x \|) \leqslant \mu(x). \tag{3.6.17}$$

Defining

$$V(t, x) = u[t, 0, \mu(x(0, t, x))],$$

it can be readily shown as in Theorem 3.6.4 that $V \in C[J \times S_\rho, R_+]$ and satisfies (2). Moreover, $V(t, 0) \equiv 0$ follows from $x(0, t, 0) \equiv 0$, $\mu(0) = 0$, and $u(t, 0, 0) \equiv 0$. From the assumption $\gamma'(u) \geqslant k > 0$, there results

$$\gamma(u_1 u_2) \geqslant k u_1 u_2 . \tag{3.6.18}$$

Furthermore, by virtue of the fact that $x = x(t, 0, x_0)$ and $x_0 = x(0, t, x)$, the inequality (3.6.15) yields

$$\beta_2^{-1}\left[\frac{\| x \|}{\sigma(t)}\right] \leqslant \| x(0, t, x)\|. \tag{3.6.19}$$

Thus, using the relations (3.6.16), (3.6.17), (3.6.18), and (3.6.19), we have

$$\begin{aligned} V(t, x) &= u[t, 0, \mu(x(0, t, x))] \\ &\geqslant \gamma[\mu(x(0, t, x))]\delta(t) \\ &\geqslant \gamma(\beta_2(\| x(0, t, x)\|))\delta(t) \\ &\geqslant \gamma\left[\beta_2\left(\beta_2^{-1}\left(\frac{\| x \|}{\sigma(t)}\right)\right)\right]\delta(t) \\ &= \gamma\left[\frac{\| x \|}{\sigma(t)}\right]\delta(t) \\ &\geqslant k\frac{\| x \|}{\sigma(t)}\,\delta(t), \end{aligned}$$

which implies, on account of the assumption $\delta(t) \geqslant k_1\sigma(t)$, that $V(t, x)$ is positive definite. The proof is complete.

Corollary 3.6.5. If $\sigma \in \mathscr{L}$ is a differentiable function for $t \in J$, then the function $g(t, u) = [\sigma'(t)/\sigma(t)]u$ is a candidate in Theorem 3.6.6.

Remark 3.6.1. Notice that, in Theorems 3.6.5 and 3.6.6, we have not assumed that the trivial solution of (3.2.3) is stable and asymptotically stable, respectively, since we do not need, in the proof, such

a specific assumption. However, these hypotheses are required to prove direct theorems. Nevertheless, the lower estimates of the solutions $u(t, 0, u_0)$ of (3.2.3) are compatible with the corresponding stability requirements. It can be seen from the proof of Theorem 3.6.4 that the lower estimate on $x(t, 0, x_0)$ and the upper estimate on $u(t, 0, u_0)$ are useful only to prove the decrescent nature of $V(t, x)$. Observe also that we need only the stability information of solutions starting at $t_0 = 0$, and this is a definite advantage.

Under the rather general assumptions of Theorem 3.6.6, it is not possible to show that $V(t, x)$ is decrescent. This can, however, be achieved in the following:

THEOREM 3.6.7. Let assumptions (i) and (iv) of Theorem 3.6.4 hold. Suppose that, in place of (3.6.15), we have

$$\beta_1 \| x_0 \|^\alpha \sigma(t) \leqslant \| x(t, 0, x_0) \| \leqslant \beta_2 \| x_0 \|^\alpha \sigma(t), \qquad t \geqslant 0,$$

where $\alpha, \beta_1, \beta_2 > 0$ are constants, and $\sigma \in \mathscr{L}$. Let the solution $u(t, 0, u_0)$ of (3.2.3) allow the estimate

$$\lambda_1 u_0 \delta(t) \leqslant u(t, 0, u_0) \leqslant \lambda_2 u_0 \delta(t), \qquad t \geqslant 0,$$

where $\lambda_1, \lambda_2 > 0$ are constants, and $\delta \in \mathscr{L}$ such that, for some constant $\beta > 0$,

$$\delta^\alpha(t) = \sigma^\beta(t).$$

Then, there exists a function $V(t, x)$ that is decrescent and that obeys (1), (2) of Theorem 3.6.6.

Proof. By choosing a continuous function $\mu(x)$ so as to satisfy

$$k_1 \| x \|^\beta \leqslant \mu(x) \leqslant k_2 \| x \|^\beta, \qquad k_1, k_2, \beta > 0,$$

and following the proof of Theorem 3.6.6 with necessary modifications, we can easily construct the proof of this theorem.

THEOREM 3.6.8. Let assumption (i) of Theorem 3.6.4 hold, and let there exist functions $\sigma_1, \sigma_2 \in \mathscr{L}$ such that

$$\beta_1 \| x_0 \| \sigma_1(t - t_0) \leqslant \| x(t, t_0, x_0) \| \leqslant \beta_2 \| x_0 \| \sigma_2(t - t_0), \qquad t \geqslant t_0, \quad (3.6.20)$$

with $\beta_1, \beta_2 > 0$ being constants and $x(t, t_0, x_0)$ being the solution of (3.2.1). Then, there exists a function $V(t, x)$ satisfying the following properties:

(1) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is positive definite, decrescent, and possesses continuous partial derivatives with respect to t and the components of x.

(2) $V'(t, x) \leqslant -\alpha V(t, x)$, $\alpha > 0$, $(t, x) \in J \times S_\rho$.

Proof. Define the function, for some fixed $T > 0$ that we choose later,

$$V(t, x) = \int_t^{t+T} \| x(s, t, x)\|^2 \, ds.$$

Because of assumption (i), one can argue, as before, to show that $V \in C[J \times S_\rho, R_+]$ and $V(t, x)$ is continuously differentiable. Furthermore, from the upper estimate of (3.6.20), we have

$$\| x(s, t, x)\| \leqslant \beta_2 \| x \| \sigma_2(0), \qquad s \geqslant t,$$

and hence

$$\begin{aligned} V(t, x) &\leqslant \int_t^{t+T} [\beta_2 \| x \| \sigma_2(0)]^2 \, ds \\ &= \beta(T) \| x \|^2, \end{aligned} \tag{3.6.21}$$

where $\beta(T) = \beta_2^2 \sigma_2^2(0) T$. Moreover, using the lower estimate of (3.6.20), it follows that

$$\begin{aligned} V(t, x) &\geqslant \int_t^{t+T} [\beta_1 \| x \| \sigma_1(s - t)]^2 \, ds \\ &= \beta_1^2 \| x \|^2 \int_0^T \sigma_1^2(u) \, du \\ &\equiv \alpha(T) \| x \|^2. \end{aligned}$$

Thus, we have shown that (1) is verified. To prove (2), observe that

$$V'(t, x) = \| x(t + T, t, x)\|^2 - \| x \|^2 + \int_t^{t+T} \frac{d}{dt} [\| x(s, t, x)\|^2] \, ds.$$

On the other hand, using relation (3.6.11), we see that

$$\begin{aligned} \frac{d}{dt} [\| x(s, t, x)\|^2] &= 2x(s, t, x) \cdot \left[\frac{\partial x(s, t, x)}{\partial t_0} + \frac{\partial x(s, t, x)}{\partial x_0} \cdot f(t, x)\right] \\ &= 0. \end{aligned}$$

Consequently,

$$\begin{aligned} V'(t, x) &= \| x(t + T, t, x)\|^2 - \| x \|^2 \\ &= [\| x(t + T, t, x)\| - \| x \|][\| x(t + T, t, x)\| + \| x \|] \\ &\leqslant [\beta_2 \| x \| \sigma_2(T) - \| x \|][\| x(t + T, t, x)\| + \| x \|]. \end{aligned}$$

We now fix T by choosing it so large that

$$\sigma_2(T) < \frac{1}{2\beta_2}.$$

This is possible, since $\sigma_2 \in \mathscr{L}$. Evidently, from this choice results the inequality

$$V'(t, x) \leqslant -\tfrac{1}{2}\| x \|^2,$$

which, in view of (3.6.21), leads to

$$\begin{aligned} V'(t, x) &\leqslant -\frac{1}{2\beta(T)} V(t, x) \\ &\equiv -\alpha V(t, x), \end{aligned}$$

setting $\alpha = 1/[2\beta(T)]$. The theorem is proved.

COROLLARY 3.6.6. Instead of the lower estimate in (3.6.20), the condition

$$\| f(t, x)\| \leqslant L(\rho)\| x \|, \qquad (t, x) \in J \times S_\rho$$

is admissible in Theorem 3.6.8.

THEOREM 3.6.9. Let the trivial solution of the system (3.2.1) be uniformly asymptotically stable. Suppose that

$$\| f(t, x_1) - f(t, x_2)\| \leqslant L(t)\| x_1 - x_2 \|$$

for $(t, x_1), (t, x_2) \in J \times S_\rho$, where $L(t) \geqslant 0$ is continuous on J, and

$$\left| \int_t^{t+\theta} L(s)\, ds \right| \leqslant K | \theta |.$$

Then, there exists a function $V(t, x)$ with the following properties:

(1) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is positive definite, decrescent, and satisfies

$$| V(t, x_1) - V(t, x_2)| \leqslant M \| x_1 - x_2 \|$$

for $(t, x_1), (t, x_2) \in J \times S_{\delta(\delta_0)}$.

(2) $D^+V(t, x) \leqslant -C(V(t, x)), (t, x) \in J \times S_\rho, C \in \mathscr{K}$.

Proof. Let us choose a function $G(r)$ such that $G(0) = 0$, $G'(0) = 0$, $G(r) > 0$, $G''(r) > 0$, and let $\alpha > 1$. Since

$$G(r) = \int_0^r du \int_0^u G''(v)\, dv$$

and

$$G\left(\frac{r}{\alpha}\right) = \int_0^{r/\alpha} du \int_0^u G''(v)\, dv,$$

we have, setting $u = w/\alpha$,

$$\begin{aligned} G\left(\frac{r}{\alpha}\right) &= \frac{1}{\alpha}\int_0^r dw \int_0^{w/\alpha} G''(v)\, dv \\ &< \frac{1}{\alpha}\int_0^r dw \int_0^w G''(v)\, dv = \frac{1}{\alpha} G(r). \end{aligned} \tag{3.6.22}$$

Define

$$V(t, x) = \sup_{\sigma \geqslant 0} G(\| x(t + \sigma, t, x)\|)\, \frac{1 + \alpha\sigma}{1 + \sigma}\,.$$

Then, for $\sigma = 0$, we obtain

$$G(\| x \|) \leqslant V(t, x). \tag{3.6.23}$$

If $\epsilon(\delta)$ is the inverse function of $\delta(\epsilon)$, we have

$$\| x(t + \sigma, t, x)\| < \epsilon(\| x \|),$$

and therefore

$$G(\| x(t + \sigma, t, x)\|) < G(\epsilon(\| x \|)).$$

Consequently, observing that $(1 + \alpha\sigma)/(1 + \sigma) < \alpha$, it follows that

$$V(t, x) \leqslant \alpha G(\epsilon(\| x \|)).$$

Since $\sigma \geqslant T(\epsilon)$ implies $\| x(t + \sigma, t, x)\| < \epsilon$, we get

$$\| x(t + \sigma, t, x)\| < \| x \|/\alpha$$

if $\sigma \geqslant T(\| x \|/\alpha)$. Thus,

$$G(\| x(t + \sigma, t, x)\|) < G(\| x \|/\alpha),$$

which, in turn, leads to

$$\begin{aligned} G(\| x(t + \sigma, t, x)\|)\, \frac{1 + \alpha\sigma}{1 + \sigma} &< \alpha G\left(\frac{\| x \|}{\alpha}\right) < G(\| x \|) \\ &\leqslant V(t, x), \end{aligned}$$

because of relations (3.6.22) and (3.6.23). This shows that

$$V(t, x) = \sup_{0 \leqslant \sigma \leqslant T(\|x\|/\alpha)} G(\| x(t + \sigma, t, x)\|)\, \frac{1 + \alpha\sigma}{1 + \sigma}\,.$$

The continuity of the function $V(t, x)$ implies that there exists a σ_1 such that

$$V(t, x) = G(\| x(t + \sigma_1, t, x)\|) \frac{1 + \alpha\sigma_1}{1 + \sigma_1}.$$

If we let $x = x(t, t_0, x_0)$, $x^* = x(t + h, t, x)$, the uniqueness of solutions of (3.2.1) shows that

$$\begin{aligned} V(t + h, x^*) &= G(\| x(t + h + \sigma^*, t + h, x^*)\|) \frac{1 + \alpha\sigma^*}{1 + \sigma^*} \\ &= G(\| x(t + h + \sigma^*, t, x)\|) \frac{1 + \alpha\sigma^*}{1 + \sigma^*}. \end{aligned}$$

Denote $\sigma^* + h = \sigma$. Then

$$\begin{aligned} \frac{1 + \alpha\sigma^*}{1 + \sigma^*} &= \frac{(1 + \alpha\sigma^*)(1 + \sigma)}{(1 + \sigma^*)(1 + \sigma)} = \frac{1 + \sigma^* + \alpha\sigma + \alpha\sigma\sigma^* - \alpha h + h}{(1 + \sigma^*)(1 + \sigma)} \\ &= \frac{(1 + \alpha\sigma)}{1 + \sigma} - \frac{\alpha h - h}{(1 + \sigma)(1 + \sigma^*)} \\ &= \frac{1 + \alpha\sigma}{1 + \sigma}\left[1 - \frac{(\alpha - 1)h}{(1 + \sigma^*)(1 + \alpha\sigma)}\right]. \end{aligned}$$

It therefore follows that

$$\begin{aligned} V(t + h, x^*) &= G(\| x(t + \sigma, t, x)\|) \frac{1 + \alpha\sigma^*}{1 + \sigma^*} \\ &= G(\| x(t + \sigma, t, x)\|) \frac{1 + \alpha\sigma}{1 + \sigma}\left[1 - \frac{(\alpha - 1)h}{(1 + \sigma^*)(1 + \alpha\sigma)}\right] \\ &\leqslant V(t, x)\left[1 - \frac{(\alpha - 1)h}{(1 + \sigma^*)(1 + \alpha\sigma)}\right], \end{aligned}$$

and hence

$$\frac{V(t + h, x^*) - V(t, x)}{h} \leqslant - \frac{(\alpha - 1)V(t, x)}{(1 + \sigma^*)(1 + \alpha\sigma)}.$$

Since

$$0 \leqslant \sigma^* < T(\| x \|/\alpha), \qquad 0 < \sigma \leqslant T(\| x^* \|/\alpha) + h,$$

using (3.6.23), it is easy to obtain

$$\begin{aligned} D^+V(t, x(t)) &= \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x(t + h, t, x)) - V(t, x)] \\ &\leqslant - \frac{(\alpha - 1)G(\| x \|)}{\{1 + T(\| x \|/\alpha)\}\{1 + \alpha T(\| x \|/\alpha)\}} \\ &\equiv - C^*(\| x \|), \qquad C^* \in \mathscr{K}, \end{aligned} \tag{3.6.24}$$

because of the fact that $\lim_{h\to 0^+} \| x^* \| = \| x \|$ and that $T(\epsilon)$ is a decreasing function.

We have seen previously that

$$V(t, x) = G(\| x(t + \sigma_1, t, x)\|) \frac{1 + \alpha\sigma_1}{1 + \sigma_1},$$

where $0 \leqslant \sigma_1 \leqslant T(\| x \|/\alpha)$. Let x_1, x_2 be such that $\| x_1 \| < \delta(\rho)$, $\| x_2 \| < \delta(\rho)$, so that the solutions $x(t, t_0, x_1)$, $x(t, t_0, x_2)$ remain in S_ρ. Let

$$r_1 = \| x(t + \sigma_1, t, x_1)\|, \qquad r_2 = \| x(t + \sigma_1, t, x_2)\|.$$

If $r_2 \geqslant r_1$, we have $G(r_2) \geqslant G(r_1)$, and hence

$$\begin{aligned} V(t, x_2) &\geqslant G(\| x(t + \sigma_1, t, x_2)\|) \frac{1 + \alpha\sigma_1}{1 + \sigma_1} \\ &\geqslant G(\| x(t + \sigma_1, t, x_1)\|) \frac{1 + \alpha\sigma_1}{1 + \sigma_1} \\ &= V(t, x_1). \end{aligned}$$

On the other hand, if $r_2 \leqslant r_1$,

$$\begin{aligned} 0 \leqslant G(r_1) - G(r_2) &= G'(\xi)(r_1 - r_2) \\ &\geqslant G'(r_1)(r_1 - r_2). \end{aligned} \tag{3.6.25}$$

Since $f(t, x)$ satisfies the Lipschitz condition, by Corollary 2.7.1, we obtain

$$\| x(t + \sigma_1, t, x_1) - x(t + \sigma_1, t, x_2)\| \leqslant \exp \left[KT\left(\frac{\| x_1 \|}{\alpha}\right)\right] \| x_1 - x_2 \|.$$

Furthermore, choosing

$$G'(r) \leqslant A \exp \left[-KT\left(\frac{\delta(r)}{\alpha}\right)\right],$$

the relation (3.6.25) leads to

$$0 \leqslant G(r_1) - G(r_2) \leqslant A \exp \left[-KT\left(\frac{\delta(r_1)}{\alpha}\right)\right] \exp \left[KT\left(\frac{\| x_1 \|}{\alpha}\right)\right] \| x_1 - x_2 \|.$$

As $\| x(t + \sigma_1, t, x_1)\| = r_1$ implies $\| x_1 \| \geqslant \delta(r_1)$, we have

$$0 \leqslant G(r_1) - G(r_2) \leqslant A\| x_1 - x_2 \|,$$

using the monotonic decreasing character of $T(\epsilon)$. It follows from these relations that

$$0 \leqslant V(t, x_1) - G(\| x(t + \sigma_1, t, x_2)\|) \frac{1 + \alpha\sigma_1}{1 + \sigma_1} \leqslant \alpha A\| x_1 - x_2 \|, \tag{3.6.26}$$

and thus that

$$V(t, x_2) \geqslant G(\| x(t + \sigma_1, t, x_2)\|) \frac{1 + \alpha\sigma_1}{1 + \sigma_1}$$
$$\geqslant V(t, x_1) - \alpha A \| x_1 - x_2 \|.$$

These considerations show that, in all cases,

$$V(t, x_2) - V(t, x_1) \geqslant - \alpha A \| x_1 - x_2 \|.$$

By interchanging the roles of x_1, x_2, we obtain

$$V(t, x_1) - V(t, x_2) \geqslant -\alpha A \| x_1 - x_2 \|,$$

and therefore there results

$$| V(t, x_1) - V(t, x_2)| \leqslant \alpha A \| x_1 - x_2 \|, \tag{3.6.27}$$

provided x_1, $x_2 \neq 0$. If $x_2 = 0$, (3.6.26) yields

$$0 \leqslant V(t, x_1) \leqslant \alpha A \| x_1 \|,$$

and hence (3.6.27) is true even when $x_2 = 0$. If $x_1 = x_2 = 0$, the relation (3.6.27) is trivially satisfied.

As previously, it is now easy to obtain (2) from relations (3.6.24) and (3.6.27) and the descrescent character of $V(t, x)$.

Finally, it remains to prove that we can choose $G(r)$ satisfying the required conditions. For this purpose, we may take

$$G(r) = A \int_0^r \exp\left[-KT\left(\frac{\delta(r)}{\alpha}\right)\right] dr.$$

One can easily verify that

$$G(0) = 0, \qquad G'(r) = A \exp[-KT(\delta(r)/\alpha) > 0, \qquad G'(0) = 0,$$

since $\delta(0) = 0$, $T(0) = \infty$, $G'(r)$ is monotone increasing, and thus $G''(r)$ exists almost everywhere and is positive. The proof is complete.

Although we have used Theorem 3.6.9 only to consider stability properties of perturbed systems, we give below a result that makes such a treatment easier.

THEOREM 3.6.10. Under the assumptions of Theorem 3.6.9, there exists a function $w(t, x)$ satisfying (1) and $D^+w(t, x) \leqslant -w(t, x)$.

Proof. By Theorem 3.6.9, there exists a function $V(t, x)$ such that (1) and (2) hold. Without loss of generality, we may assume that $C(u)$ is differentiable, $C'(0) = 0$, and $\int_0 ds/C(s) = \infty$. If $C(u)$ does not have these properties, we can choose such a function $C_1(u)$ satisfying $C_1(u) \leqslant C(u)$. Consider the function

$$\lambda(u) = \exp\left[\int_\rho^u \frac{ds}{C(s)}\right] \quad \text{for} \quad 0 < u \leqslant \rho.$$

It is clear from the properties of $C(u)$ that $\lambda(0) = 0$ and $\lambda'(u)$ exists and is continuous on $0 \leqslant u \leqslant \rho$, so that $|\lambda'(u)| \leqslant K$. We now definc the desired function $w(t, x)$ by

$$w(t, x) = \lambda(V(t, x)).$$

It is easy to check that this function verifies the required properties. The proof is therefore complete.

3.7. Stability by the first approximation

Let $x_0(t)$ be a solution of (3.2.1). Set $y = x - x_0(t)$, and obtain the equation

$$\begin{aligned} y' &= x' - x_0'(t) = f(t, x) - f(t, x_0(t)) \\ &= f(t, y + x_0(t)) - f(t, x_0(t)) \\ &= \frac{\partial f(t, x_0(t))}{\partial x} y + 0(\|y\|). \end{aligned}$$

A natural question is whether we can legitimately neglect the terms of the form $0(\|y\|)$. In the theory of stability by the first approximation, this procedure is justified. The following theorem is to that effect.

THEOREM 3.7.1. Suppose that the trivial solution $x = 0$ of (3.2.1) is exponentially asymptotically stable and $f(t, x)$ in (3.2.1) is linear in x. Assume further that the function $R(t, x)$ in (3.5.1) satisfies the relation

$$\| R(t, x)\| \leqslant C\| x \| \tag{3.7.1}$$

for $\| x \| \leqslant \rho$, C being a sufficiently small constant. Then, the trivial solution $x = 0$ of (3.5.1) is exponentially asymptotically stable.

Proof. By Theorem 3.6.1, there exists a function $V(t, x)$ having the following properties:

(i) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is Lipschitzian in x for a constant $K > 0$.

(ii) $\| x \| \leqslant V(t, x) \leqslant K \| x \|, (t, x) \in J \times S_\rho$.

(iii) $D^+V(t, x)_{(3.2.1)} \leqslant -\alpha V(t, x), \alpha > 0, (t, x) \in J \times S_\rho$.

Hence, for $\| x \| < \rho$, we have

$$D^+V(t, x)_{(3.5.1)} \leqslant D^+V(t, x)_{(3.2.1)} + K\| R(t, x)\|, \tag{3.7.2}$$

using the fact that $V(t, x)$ is Lipschitzian with a constant K. Define $m(t) = V(t, x(t))$, where $x(t)$ is any solution of (3.5.1) such that $\| x_0 \| < \frac{1}{2}\rho/K$. Because of (ii), whenever $\| x_0 \| < \frac{1}{2}\rho/K$, we have $m(t_0) < \frac{1}{2}\rho$. We claim that $m(t) < \rho$ for $t \geqslant t_0$. If this is not true, there exist numbers t_1 and t_2 such that

$$m(t_2) = \tfrac{1}{2}\rho, \qquad m(t_1) = \rho,$$

and

$$m(t) \geqslant \tfrac{1}{2}\rho, \qquad t \in [t_2, t_1].$$

Thus, $D^+m(t_2) \geqslant 0$. On the other hand, since $m(t) \leqslant \rho$ for $t_0 \leqslant t \leqslant t_1$, we have, by (ii), that

$$\| x(t)\| \leqslant \rho, \qquad t_0 \leqslant t \leqslant t_1 .$$

Hence, using condition (iii), together with (3.7.1) and (3.7.2), leads to

$$\begin{aligned} D^+m(t_2) &\leqslant -\alpha m(t_2) + KC\| x(t_2)\| \\ &= m(t_2)[-\alpha + KC], \end{aligned}$$

because $\| x \| \leqslant V(t, x)$. Since C is sufficiently small, there exists a $\gamma > 0$ such that $C \leqslant (\alpha - \gamma)/K$. This implies that $D^+m(t_2) < 0$, and this contradiction proves that $m(t) < \rho$ for $t \geqslant t_0$. Consequently, $\| x(t)\| < \rho$ for $t \geqslant t_0$. Thus, whenever $\| x_0 \| < \frac{1}{2}\rho/K$, we have

$$D^+V(t, x(t)) \leqslant -\gamma V(t, x(t)),$$

and therefore, by Theorem 1.4.1,

$$V(t, x(t)) \leqslant V(t_0, x_0) \exp[-\gamma(t - t_0)], \qquad t \geqslant t_0 .$$

It is easy to obtain from this inequality a further inequality

$$\| x(t, t_0, x_0)\| \leqslant \rho \exp[-\gamma(t - t_0)], \qquad t \geqslant t_0 ,$$

which proves the stated result. The proof of the theorem is complete.

The linearity of $f(t, x)$ in x can be dropped, if $f(t, x)$ satisfies a Lipschitz condition in x for a constant $L = L(\rho) > 0$. The next theorem is therefore a generalization of Theorem 3.7.1.

THEOREM 3.7.2. Assume that the trivial solution $x = 0$ of (3.2.1) is exponentially asymptotically stable and $f(t, x)$ satisfies a Lipschitz condition for a constant $L = L(\rho) > 0$. Let

$$\| R(t, x)\| \leqslant N\| x \| \qquad \text{for} \quad \| x \| \leqslant \rho,$$

where $N = N(\rho)$ is sufficiently small. Then, the trivial solution of (3.5.1) is exponentially asymptotically stable.

Proof. By Theorem 3.6.2, there exists a function $V(t, x)$ with the following properties:

(i) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ satisfies a Lipschitz condition with $M = K^{[L+(1-q)\alpha]/q\alpha}$, $0 < q < 1$.

(ii) $\| x \| \leqslant V(t, x) \leqslant K \| x \|$, $(t, x) \in J \times S_\rho$.

(iii) $D^+V(t, x) \leqslant -q\alpha V(t, x)$.

Now, following the proof of Theorem 3.7.1, one can prove the stated result. Here we have to choose $\gamma > 0$ such that $N \leqslant (\alpha q - \gamma)/M$.

THEOREM 3.7.3. Assume that the trivial solution of (3.2.1) is exponentially asymptotically stable and that $f(t, x)$ is linear in x. Suppose further that

(i) $F \in C[J \times S_\rho, R^n]$, and, given any $\epsilon > 0$, there exist $\delta(\epsilon)$, $T(\epsilon)$ such that

$$\|F(t, x)\| \leqslant \epsilon\| x \|, \qquad \| x \| \leqslant \delta(\epsilon), \quad t \geqslant T(\epsilon);$$

(ii) $R \in C[J \times S_\rho, R^n]$, $R(t, 0) \equiv 0$, and there exists an $\eta > 0$ such that, if $\| x \| \leqslant \eta$,

$$\| R(t, x)\| \leqslant \gamma(t), \qquad t \geqslant 0,$$

where $\gamma \in C[J, R_+]$ and

$$\int_t^{t+1} \gamma(s)\, ds \to 0 \qquad \text{as} \quad t \to \infty.$$

Then, there exists a $T_0 \geqslant 0$ such that, for $t_0 \geqslant T_0$, the trivial solution of

$$x' = f(t, x) + F(t, x) + R(t, x) \tag{3.7.3}$$

is asymptotically stable.

Proof. Since the solution $x = 0$ of (3.2.1) is assumed to be exponentially asymptotically stable, there exists, by Theorem 3.6.1, a function $V(t, x)$ satisfying

(a) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is Lipschitzian in x for a constant $K > 0$;

(b) $\| x \| \leqslant V(t, x) \leqslant K \| x \|$, $(t, x) \in J \times S_\rho$;

(c) $D^+V(t, x)_{(3.2.1)} \leqslant -\alpha V(t, x)$, $\alpha > 0$, $(t, x) \in J \times S_\rho$.

Let ϵ be given such that $0 < \epsilon < \min(\alpha/K, \eta)$. Choose $T_0 \geqslant 1$ so large that, for $t \geqslant T_0$, we have

$$K \int_1^t \exp[-(\alpha - K\epsilon)(t - s)]\gamma(s)\, ds < \tfrac{1}{2}\delta(\epsilon) = \delta_1, \tag{3.7.4}$$

where $\delta(\epsilon) \leqslant \epsilon$. As shown in Theorem 2.14.6, this choice is possible. If $\| x \| < \rho$, it is easy to obtain

$$D^+V(t, x)_{(3.7.3)} \leqslant D^+V(t, x)_{(3.2.1)} + K[\| F(t, x)\| + \| R(t, x)\|]. \tag{3.7.5}$$

Consider the function $m(t) = V(t, x(t))$, $x(t) = x(t, t_0, x_0)$ being any solution of (3.7.3). We maintain that, whenever $\| x_0 \| < \delta_1/K$, we have $\| x(t)\| < \delta(\epsilon)$, $t \geqslant t_0$. If this were false, there would exist a $t_1 > t_0 \geqslant T_0$ such that

$$\| x(t_1)\| = \delta(\epsilon), \quad \| x(t)\| \leqslant \delta(\epsilon), \qquad t \in [t_0, t_1].$$

In view of conditions (i) and (ii) and (3.7.5), there results the differential inequality

$$D^+m(t) \leqslant -(\alpha - K\epsilon)m(t) + K\gamma(t), \qquad t \in [t_0, t_1].$$

Here, we have used that $\| x(t)\| \leqslant V(t, x(t))$ and $D^+V(t, x)_{(3.2.1)} \leqslant -\alpha V(t, x)$. According to Theorem 1.4.1, we can deduce

$$\begin{aligned} m(t) \leqslant{}& m(t_0) \exp[-(\alpha - K\epsilon)(t - t_0)] \\ &+ K \int_{t_0}^t \gamma(s) \exp[-(\alpha - K\epsilon)(t - s)]\, ds \end{aligned}$$

for $t \in [t_0, t_1]$, which, in turn, gives, for $t \in [t_0, t_1]$,

$$\begin{aligned} \| x(t)\| \leqslant{}& K\| x_0 \| \exp[-(\alpha - K\epsilon)(t - t_0)] \\ &+ K \int_1^t \gamma(s) \exp[-(\alpha - K\epsilon)(t - s)]\, ds, \end{aligned} \tag{3.7.6}$$

using relation (b). At $t = t_1$, we shall then have a contradiction

$$\delta(\epsilon) = \| x(t_1)\| < \delta_1 + \delta_1 = \delta(\epsilon).$$

Thus, $\| x_0 \| < \delta_1/K$ implies $\| x(t)\| < \delta(\epsilon)$, $t \geqslant t_0 \geqslant T_0$. Consequently, (3.7.6) is valid for all $t \geqslant t_0$, and the asymptotic stability of the trivial solution of (3.7.3) follows, as in Theorem 2.14.4.

THEOREM 3.7.4. Assume that $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, $\partial f(t, x)/\partial x$ exists and is continuous for $(t, x) \in J \times S_\rho$, and that the trivial solution of the variational system

$$x' = f_x(t, 0)x \tag{3.7.7}$$

is exponentially asymptotically stable. Suppose that assumption (ii) of Theorem 3.7.3 holds. Then, there exists a $T_0 \geqslant 0$ such that, for $t_0 \geqslant T_0$, the trivial solution of (3.5.1) is asymptotically stable.

Proof. Since $f(t, 0) \equiv 0$ and $\partial f(t, x)/\partial x$ exists and is continuous, we have

$$f(t, x) = f_x(t, 0)x + F(t, x),$$

where $F(t, x)$ satisfies assumption (i) of Theorem 3.7.3. Hence, the differential system (3.5.1) takes the form

$$x' = f_x(t, 0)x + F(t, x) + R(t, x).$$

It is therefore clear that the stated result follows by Theorem 3.7.3.

THEOREM 3.7.5. Let us suppose that the trivial solution of (3.2.1) is generalized exponentially asymptotically stable and that $f(t, x)$ is linear in x. Suppose further that the perturbation $R(t, x)$ verifies the estimate

$$\| R(t, x)\| \leqslant w(t, \| x \|), \qquad (t, x) \in J \times S_\rho,$$

where $w \in C[J \times R_+, R_+]$, $w(t, 0) \equiv 0$, and $w(t, u)$ is nondecreasing in u for $t \in J$. Then, the stability or asymptotic stability of the trivial solution of

$$u' = -p'(t)u + K(t)w(t, u), \qquad u(t_0) = u_0 \geqslant 0,$$

implies the equistability or equi-asymptotic stability of the trivial solution of the perturbed system (3.5.1).

Proof. On the basis of Theorem 3.6.1, there exists a function $V(t, x)$ fulfilling the following conditions:

(i) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is Lipschitzian in x for a function $K(t) \geqslant 0$.

(ii) $\| x \| \leqslant V(t, x) \leqslant K(t) \| x \|$, $(t, x) \in J \times S_\rho$.

(iii) $D^+ V(t, x)_{(3.2.1)} \leqslant -p'(t) V(t, x)$, $(t, x) \in J \times S_\rho$.

Thus, whenever $\| x \| < \rho$, it can be readily verified that

$$D^+ V(t, x)_{(3.5.1)} \leqslant D^+ V(t, x)_{(3.2.1)} + K(t) \| R(t, x) \|,$$

which yields a further inequality

$$D^+ V(t, x)_{(3.5.1)} \leqslant -p'(t) V(t, x) + K(t) w(t, V(t, x))$$

because of (ii), (iii), and the monotonic character of $w(t, u)$ in u. We can now apply Theorems 3.3.1, 3.3.2, 3.4.1, and 3.4.2 to obtain the desired result.

THEOREM 3.7.6. Let $f \in C[S_\rho, R^n]$, $f(0) = 0$, $f(\alpha x) = \alpha^m f(x)$, $m > 1$, and the trivial solution of

$$x' = f(x) \tag{3.7.8}$$

be asymptotically stable. Then, the trivial solution of the system

$$dy/ds = F(y), \tag{3.7.9}$$

where

$$F(y) = \begin{cases} \dfrac{f(y)}{\| y \|^{m-1}}, & \| y \| \neq 0 \\ 0, & \| y \| = 0, \end{cases}$$

is exponentially asymptotically stable.

Proof. Let $y(s, s_0, x_0)$ be a solution of (3.7.9) and

$$t(s) = \int_0^s \frac{du}{\| y(u, s_0, x_0) \|^{m-1}},$$

$s(t)$ being the inverse function of $t(s)$. Set $x(t) = y(s(t), s_0, x_0)$. Then,

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{d}{ds}(y(s(t), s_0, x_0)) \frac{ds(t)}{dt} \\ &= \frac{f[y(s(t), s_0, x_0)]}{\| y(s(t), s_0, x_0) \|^{m-1}} \left(\frac{dt(s)}{ds} \right)^{-1} \\ &= f(x(t)). \end{aligned}$$

Furthermore, $x(t_0) = x_0$, where $t_0 = t(s_0)$. Since the solution $x = 0$ of (3.7.8) is assumed to be asymptotically stable, the function $f(x)$ being autonomous, we have

$$\| x(t)\| \leqslant a(\| x_0 \|)\sigma(t - t_0), \qquad t \geqslant t_0,$$

where $a \in \mathscr{K}$, $\sigma \in \mathscr{L}$. It therefore follows that

$$\begin{aligned}\| y(s, s_0, x_0)\| &\leqslant a(\| x_0 \|)\sigma[t(s) - t(s_0)] \\ &\leqslant a(\| x_0 \|)\sigma \left[\int_{s_0}^{s} \frac{du}{\| y(u, s_0, x_0)\|^{m-1}}\right] \\ &\leqslant a(\| x_0 \|)\sigma[\beta(s - s_0)] \\ &= a(\| x_0 \|)\sigma_1(s - s_0), \end{aligned} \tag{3.7.10}$$

where $\sigma_1 \in \mathscr{L}$, using the fact that $y(u, s_0, x_0)$ is bounded, and so

$$\frac{1}{\| y(u, s_0, x_0)\|^{m-1}} \geqslant \beta > 0.$$

From the evaluation (3.7.10), the uniform asymptotic stability of the solution $y = 0$ of (3.7.9) is evident.

Clearly, $F(y)$ is homogeneous in y of first degree. Hence, because of uniqueness of solutions, it results that

$$y(s, s_0, \alpha x_0) = \alpha y(s, s_0, x_0).$$

Moreover, using (3.7.10), we derive that

$$\begin{aligned}\| y(s, s_0, x_0)\| &= \left\| y\left(s, s_0, \frac{\| x_0 \|}{\delta_0} \frac{x_0}{\| x_0 \|} \delta_0\right)\right\| \\ &\leqslant \frac{\| x_0 \|}{\delta_0} a(\delta_0)\sigma_1(s - s_0) \\ &= a^*(\| x_0 \|)\sigma_1(s - s_0),\end{aligned}$$

which implies that $a^*(u)$ is linear in u.

One can now conclude, on the basis of Corollary 3.6.6 and the facts that $a^*(u)$ is linear in u and $F(y)$ is homogeneous in y of first degree, that the trivial solution of (3.7.9) is exponentially asymptotically stable.

THEOREM 3.7.7. Let $f \in C[S_\rho, R^n]$, $f(0) = 0$, $\alpha^m f(x) = f(\alpha x)$, $m > 1$, and the trivial solution of (3.7.8) be asymptotically stable. Assume that $R \in C[J \times S_\rho, R^n]$ and

$$\| R(t, x)\| \leqslant C\| x \|^m, \qquad (t, x) \in J \times S_\rho, \tag{3.7.11}$$

C being a sufficiently small constant. Then, the trivial solution of the system

$$x' = f(x) + R(t, x) \tag{3.7.12}$$

is uniformly asymptotically stable.

Proof. Let $x(t, t_0, x_0)$ be a solution of (3.7.12). Define

$$s(t) = \int_0^t \| x(u, t_0, x_0)\|^{m-1}\, du,$$

and let $t(s)$ be the inverse function of $s(t)$. Setting $y(t) = x(t(s), t_0, x_0)$, it is easy to check that

$$\frac{dy(s)}{ds} = \frac{f(y(s))}{\| y(s)\|^{m-1}} + \frac{R(t(s), y(s))}{\| y(s)\|^{m-1}},$$

verifying that $y(s)$ satisfies the system

$$dy/ds = F(y) + R^*(s, y),$$

where, in view of (3.7.11),

$$\begin{aligned} \| R^*(s, y)\| &= \frac{\| R(t(s), y)\|}{\| y \|^{m-1}} \\ &\leqslant C\| y \|. \end{aligned}$$

The conditions of Theorem 3.7.1 being fulfilled, it follows that

$$\| y(s)\| \leqslant K\| x_0 \| \exp[-\alpha(s - s_0)] \qquad (s_0 = s(t_0)),$$

whence

$$\| x(t(s), t_0, x_0)\| \leqslant K\| x_0 \| \exp[-\alpha(s - s_0)],$$

and therefore

$$\begin{aligned} \| x(t, t_0, x_0)\| &\leqslant K\| x_0 \| \exp[-\alpha(s(t) - s(t_0))] \\ &\leqslant K\| x_0 \| \exp\left[-\alpha \int_{t_0}^t \| x(u, t_0, x_0)\|^{m-1}\, du\right]. \end{aligned} \tag{3.7.13}$$

Since the solution $y(s)$ is defined for all $s \geqslant s_0$, $\lim_{t\to\infty} s(t) = \infty$. This shows that the integral in (3.7.13) is divergent, proving the exponential asymptotic stability of the trivial solution of (3.7.12). The theorem is proved.

The next theorem is of less general character, which may prove effective in certain concrete cases. The importance of the theorem, however, is that a judicious selection of $V(t, x)$, reflecting more closely particular properties of the given system, frequently leads to much more precise results rather than yielding to the temptation of choosing $V(t, x)$ as simple as possible, such as $V(t, x) = \| x \|$.

THEOREM 3.7.8. Let the following assymptions hold:

(i) There exists a continuously differentiable matrix $G(t)$, which is self-adjoint and positive, that is, the Hermitian form (Gx, x) is positive definite, and $\lambda_1, \lambda_2 > 0$ are the smallest and the largest eigenvalues of $G(t)$.

(ii) The function $q \in C[J, R]$ is the largest eigenvalue of the matrix $G^{-1}(t)\, Q(t)$, where

$$Q(t) = \frac{dG(t)}{dt} + G(t)A(t) + A^*(t)G(t),$$

$A(t)$ being a continuous matrix on J and $A^*(t)$ its transpose.

(iii) $R \in C[J \times S_\rho, R^n]$, and

$$\| R(t, x)\| \leqslant \beta(t)\| x \|^\alpha, \qquad 0 < \alpha \leqslant 1,$$

where $\beta \in C[J, R_+]$.

Then, the stability properties of the trivial solution of

$$u' = q(t)u + 2\beta(t)(\lambda_2 \lambda_1^{-\alpha})^{1/2} u^{(1+\alpha)/2} \tag{3.7.14}$$

imply the corresponding stability properties of the trivial solution of

$$x' = A(t)x + R(t, x). \tag{3.7.15}$$

In particular, if $\alpha = 1$ and

$$\int_{t_0}^{\infty} [q(s) + 2(\lambda_2 \lambda_1^{-1})^{1/2} \beta(s)]\, ds = -\infty, \tag{3.7.16}$$

then the trivial solution of (3.7.15) is asymptotically stable.

Proof. Let us consider the Lyapunov function defined by

$$V(t, x) = (G(t)x, x).$$

We then have

$$V'(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} [A(t)x + R(t, x)]$$
$$= (G'x, x) + (GAx, x) + (Gx, Ax) + (GR, x) + (Gx, R)$$
$$= (G'x, x) + (GAx, x) + (A^*Gx, x) + (GR, x) + (Gx, R)$$
$$= (Q(t)x, x) + (GR, x) + (Gx, R).$$

In view of the definition of $Q(t)$, we also obtain

$$G^{-1}(t)Q(t) = G^{-1} \frac{dG}{dt} + A + G^{-1}A^*G,$$

and hence

$$(Qx, x) \leqslant q(t)(Gx, x) = q(t)V(t, x).$$

Furthermore, it is easy to infer that

$$(Gx, R) + (GR, x) \leqslant 2[(Gx, x)(GR, R)]^{1/2}.$$

Since

$$\lambda_1(x, x) \leqslant (Gx, x) \leqslant \lambda_2(x, x),$$

it follows, on account of (iii), that

$$(GR, R) \leqslant \lambda_2(R, R) \leqslant \lambda_2\beta^2(t)(x, x)^\alpha$$
$$= \lambda_2\lambda_1^{-\alpha}\beta^2(t)\lambda_1^{\alpha}(x, x)^\alpha$$
$$\leqslant \lambda_2\lambda_1^{-\alpha}\beta^2(t)[V(t, x)]^\alpha.$$

Taking into account all of these inequalities, we arrive at

$$V'(t, x) \leqslant q(t)V(t, x) + 2\beta(t)(\lambda_2\lambda_1^{-\alpha})^{1/2}[V(t, x)]^{(1+\alpha)/2}.$$

We can now use Theorems 3.3.3, 3.3.4, 3.4.3, and 3.4.4 to get the stated result.

If $\alpha = 1$ and (3.7.16) holds, it is clear that the trivial solution of (3.7.14) is asymptotically stable, and consequently the asymptotic stability of the trivial solution of (3.7.15) follows.

3.8. Total stability

The theorems in this section emphasize the importance of uniform asymptotic stability in that, if the trivial solution is uniformly asymp-

totically stable, it also has certain stability properties under different classes of perturbations.

DEFINITION 3.8.1. The trivial solution $x = 0$ of (3.2.1) is said to be T_1-*totally stable* (stable with respect to permanent perturbations) if, for every $\epsilon > 0$, $t_0 \in J$, there exist two positive numbers $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ such that for every solution $x(t, t_0, x_0)$ of the perturbed differential equation (3.5.1) the inequality

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0$$

holds, provided that

$$\| x_0 \| < \delta_1$$

and

$$\| R(t, x)\| < \delta_2 \qquad \text{for} \quad \| x \| < \epsilon,\ t \in J. \tag{3.8.1}$$

THEOREM 3.8.1. If the trivial solution $x = 0$ of (3.2.1) is uniformly asymptotically stable and

$$\|f(t, x) - f(t, y)\| \leqslant L(t)\| x - y \|$$

for $(t, x), (t, y) \in J \times S_\rho$ and

$$\left| \int_t^{t+u} L(s)\, ds \right| \leqslant K| u |,$$

then the trivial solution $x = 0$ of (3.2.1) is also T_1-totally stable.

Proof. According to Theorem 3.6.9, it follows from the uniform asymptotic stability of the trivial solution of (3.2.1) that there exists a function $V(t, x)$ satisfying the following conditions:

(a) $b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|)$, $a, b \in \mathscr{K}$.

(b) $| V(t, x) - V(t, y)| \leqslant M \| x - y \|$, $(t, x), (t, y) \in J \times S_{\delta(\delta_0)}$, where $\delta(\epsilon)$ and δ_0 appear in (S_6).

(c) $D^+V(t, x) \leqslant -C[V(t, x)]$, $C \in \mathscr{K}$.

Let $0 < \epsilon < \delta(\delta_0)$ be given. Choose $\delta_1 = \delta_1(\epsilon)$ such that

$$b(\epsilon) > a(\delta_1). \tag{3.8.2}$$

Define $m(t) = V(t, x(t))$, where $x(t) = x(t, t_0, x_0)$ is any solution of (3.5.1) such that $\| x_0 \| < \delta_1$. This, because of (a), implies $m(t_0) < a(\delta_1)$. We claim that

$$m(t) < b(\epsilon), \qquad t \geqslant t_0. \tag{3.8.3}$$

If this is false, there exist two numbers $t_1 > t_2 > t_0$ such that

$$m(t_2) = a(\delta_1), \qquad m(t_1) = b(\epsilon)$$

and

$$m(t) \geqslant a(\delta_1), \qquad t_2 \leqslant t \leqslant t_1 .$$

Thus, we would have

$$D^+m(t_2) \geqslant 0. \tag{3.8.4}$$

On the other hand, observe that, for $t_0 \leqslant t \leqslant t_1$, we have

$$b(\| x(t)\|) \leqslant V(t, x(t)) = m(t) \leqslant b(\epsilon),$$

which implies that

$$\| x(t)\| \leqslant \epsilon < \delta(\delta_0), \qquad t \in [t_0, t_1].$$

Thus, we can use conditions (b) and (c) to arrive at

$$D^+m(t_2) \leqslant -C[a(\delta_1)] + M\| R(t_2, x(t_2))\|. \tag{3.8.5}$$

Now choose

$$\delta_2 = \delta_2(\epsilon) = \frac{C[a(\delta_1(\epsilon))]}{M}.$$

The fact that $\| x(t_2)\| \leqslant \epsilon$, together with relations (3.8.1) and (3.8.5), leads to

$$D^+m(t_2) < -C[a(\delta_1)] + M\delta_2 = 0,$$

which contradicts (3.8.4) and proves (3.8.3). From this follows that the trivial solution $x = 0$ of (3.2.1) is T_1-totally stable. The theorem is proved.

A variant of the notion of total stability with respect to perturbations may be defined if, instead of (3.8.1), we only require that the perturbations be bounded in the mean.

DEFINITION 3.8.2. The trivial solution $x = 0$ of (3.2.1) is said to be T_2-totally stable (stable under permanent perturbations bounded in the mean) if, for every $\epsilon > 0$, $t_0 \in J$, and $T > 0$, there exist two positive numbers $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ such that, for every solution $x(t, t_0, x_0)$ of the perturbed differential system (3.5.1), the inequality

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0$$

holds, provided that

$$\| x_0 \| < \delta_1 , \qquad \| R(t, x)\| \leqslant \lambda(t) \qquad \text{for} \quad \| x \| \leqslant \epsilon \tag{3.8.6}$$

and

$$\int_t^{t+T} \lambda(s)\, ds < \delta_2 . \tag{3.8.7}$$

THEOREM 3.8.2. Under the assumptions of Theorem 3.8.1, the trivial solution $x = 0$ of (3.2.1) is T_2-totally stable.

Proof. We proceed as in the proof of Theorem 3.8.1 and choose $\delta_1 = \delta_1(\epsilon)$ by relation (3.8.2). Let $\| x_0 \| < \delta_1$ and $m(t) = V(t, x(t))$, where $x(t) = x(t, t_0, x_0)$ is any solution of (3.5.1). As before, $m(t_0) < a(\delta_1)$, and the claim (3.8.3) is true. If it is not the case, there exists a $t_1 > t_0$ such that $m(t) \leqslant b(\epsilon)$ for $t_0 \leqslant t \leqslant t_1$, which implies that

$$\| x(t)\| \leqslant \epsilon < \delta(\delta_0), \qquad t_0 \leqslant t \leqslant t_1 .$$

Denote $t_1 - t_0 = T$, and choose

$$\delta_2 = \delta_2(\epsilon) < b(\epsilon) - J^{-1}[J(a(\delta_1))]/M, \tag{3.8.8}$$

where

$$J(u) - J(u_0) = \int_{u_0}^{u} \frac{ds}{C(s)}$$

and J^{-1} is the inverse function of J. From relations (b) and (c), we obtain, for $t \in [t_0 , t_1]$,

$$D^+V(t, x(t)) \leqslant -C[V(t, x(t))] + M\| R(t, x(t))\|.$$

If we now define

$$z(t) = V(t, x(t)) - v(t),$$

where

$$v(t) = M \int_{t_0}^{t} \| R(s, x(s))\|\, ds,$$

we obtain the inequality

$$D^+z(t) \leqslant -C[z(t)],$$

using the monotonic character of $C(u)$ and the fact that $z(t) \geqslant V(t, x(t))$, which implies, in view of Theorem 1.4.1, that

$$z(t) \leqslant J^{-1}[J(V(t_0 , x_0)] - (t - t_0)], \qquad t \in [t_0 , t_1].$$

Note that the maximal solution of $u' = -C(u), u(t_0) = V(t_0, x_0)$ is just the right-hand side of the foregoing inequality. Thus, it follows that

$$V(t, x(t)) \leqslant J^{-1}[J[V(t_0, x_0)] - (t - t_0)] + v(t), \qquad t \in [t_0, t_1].$$

From this, we derive a further inequality, using the facts that $\| x(t)\| \leqslant \epsilon$ for $t_0 \leqslant t \leqslant t_0 + T$, $V(t_0, x_0) < a(\delta_1)$ and relations (3.8.6), (3.8.7), and (3.8.8),

$$b(\epsilon) \leqslant V(t_0 + T, x(t_0 + T)) \leqslant J^{-1}[J[a(\delta_1)] - T] + M\delta_2 < b(\epsilon).$$

This contradiction assures that $m(t) < b(\epsilon)$, $t \geqslant t_0$, which, in its turn, implies T_2-total stability of the solution $x = 0$ of (3.2.1). This completes the proof.

In the case of certain perturbations that approach zero as $t \to \infty$, uniform asymptotic stability implies total asymptotic stability defined below.

Definition 3.8.3. The trivial solution $x = 0$ of (3.2.1) is said to be *totally uniformly asymptotically stable* if, for solutions $x(t, t_0, x_0)$ of the perturbed differential system (3.5.1), the definition (S_6) holds, provided that $R(t, 0) \equiv 0$ and

$$\| R(t, x)\| \leqslant \sigma(t), \qquad \sigma \in \mathscr{L}, \tag{3.8.9}$$

uniformly for $\| x \| < \rho$.

Theorem 3.8.3. Under the assumptions of Theorem 3.8.1, the trivial solution $x = 0$ of (3.2.1) is totally uniformly asymptotically stable, provided $R(t, x)$ also satisfies a Lipschitz condition in x.

Proof. Consider the same function $V(t, x)$ as in Theorem 3.8.1. If $\| x \| < \delta(\delta_0) = \rho_0$, we would have

$$\begin{aligned} D^+ V(t, x)_{(3.5.1)} &\leqslant -C[V(t, x)] + M\| R(t, x)\| \\ &\leqslant -C[V(t, x)] + M\sigma(t) \\ &\equiv g(t, V(t, x)). \end{aligned}$$

Let $0 < \alpha \leqslant \beta < \rho_0$ be given, and let $K(\alpha, \beta) = \frac{1}{2}C(\alpha)$. Since $\sigma \in \mathscr{L}$, there exists a $\theta(\alpha, \beta) \geqslant 0$ such that $\sigma(t) \leqslant \frac{1}{2}C(\alpha)/M$, $t > \theta(\alpha, \beta)$. Thus, if $\alpha \leqslant u \leqslant \beta$, $t \geqslant \theta(\alpha, \beta)$, we have

$$\begin{aligned} g(t, u) &= -C(u) + M\sigma(t) \\ &\leqslant -C(\alpha) + \tfrac{1}{2}C(\alpha) = -K(\alpha, \beta). \end{aligned}$$

The conditions of Theorem 3.4.10 being verified, it is easy to see that Theorem 3.8.3 is proved.

3.9. Integral stability

We shall continue to study the system (3.2.1) and its perturbed system (3.5.1). However, for the purposes of this section, it becomes necessary to assume that $\rho = \infty$ so that functions f, R occurring therein are such that $f, R \in C[J \times R^n, R^n]$.

DEFINITION 3.9.1. The trivial solution $x = 0$ of (3.2.1) is said to be

(I_1) *Equi-integrally stable* if, for every $\alpha \geqslant 0$ and $t_0 \in J$, there exists a positive function $\beta = \beta(t_0, \alpha)$, which is continuous in t_0 for each α and $\beta \in \mathscr{K}$ for each t_0, such that, for every solution $x(t, t_0, x_0)$ of the perturbed differential system (3.5.1), the inequality

$$\| x(t, t_0, x_0)\| < \beta, \qquad t \geqslant t_0,$$

holds, provided that

$$\| x_0 \| \leqslant \alpha,$$

and, for every $T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{\|x\| \leqslant \beta} \| R(s\ x)\| \, ds \leqslant \alpha.$$

(I_2) *Uniformly-integrally stable* if the β in (I_1) is independent of t_0.

(I_3) *Equi-asymptotically integrally stable* if (I_1) holds and, for every $\epsilon > 0$, $\alpha \geqslant 0$, and $t_0 \in J$, there exist positive numbers $T = T(t_0, \alpha, \epsilon)$ and $\gamma = \gamma(t_0, \alpha, \epsilon)$ such that, for every solution of the system (3.5.1), the inequality

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0 + T,$$

holds, provided that

$$\| x_0 \| \leqslant \alpha$$

and

$$\int_{t_0}^{\infty} \sup_{\|x\| \leqslant \beta} \| R(s, x)\| \, ds < \gamma.$$

(I_4) *Uniformly-asymptotically integrally stable* if the T and γ in (I_3) are independent of t_0 and (I_2) holds.

In addition to the scalar differential equation (3.2.3), let us consider the following perturbed equation:

$$u' = g(t, u) + \phi(t), \qquad u(t_0) = u_0, \tag{3.9.1}$$

where $g \in C[J \times R_+, R]$ and $\phi \in C[J, R_+]$.

DEFINITION 3.9.2. The null solution $u = 0$ of (3.2.3) is said to be

(I_1^*) *Equi-integrally stable* if, for every $\alpha_1 \geqslant 0$, $t_0 \in J$, there exists a positive function $\beta_1 = \beta_1(t_0, \alpha_1)$ that is continuous in t_0 for each α_1 and $\beta_1 \in \mathscr{K}$ for each t_0 such that, whichever be the function $\phi \in C[J, R_+]$ with

$$\int_{t_0}^{t_0+T} \phi(s)\, ds \leqslant \alpha_1$$

for every $T > 0$, every solution $u(t, t_0, u_0)$ of the perturbed scalar equation (3.9.1) satisfies the inequality

$$u(t, t_0, u_0) < \beta_1, \qquad t \geqslant t_0,$$

provided that

$$u_0 \leqslant \alpha_1.$$

The definitions (I_2^*)–(I_4^*) may be formulated similarly.

THEOREM 3.9.1. Assume that there exist functions $V(t, x)$ and $g(t, u)$ satisfying the following properties:

(i) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$.

(ii) $V \in C[J \times R^n, R_+]$, $V(t, x)$ is Lipschitzian in x for a constant $M > 0$, and there exists a function $b \in \mathscr{K}$ such that $b(u) \to \infty$ as $u \to \infty$, and

$$b(\| x \|) \leqslant V(t, x), \qquad (t, x) \in J \times R^n.$$

(iii) $D^+V(t, x)_{(3.2.1)} \leqslant g(t, V(t, x))$, $(t, x) \in J \times R^n$.

Then, the equi-integral stability of the null solution $u = 0$ of (3.2.3) implies the equi-integral stability of the trivial solution $x = 0$ of (3.2.1).

Proof. Let $\alpha \geqslant 0$ and $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \alpha$. Since $V(t, x)$ is Lipschitzian in x for a constant $M > 0$, we have

$$| V(t, x) - V(t, y)| \leqslant M\| x - y \|, \tag{3.9.2}$$

from which it follows that $V(t_0, x_0) \leqslant M\alpha \equiv \alpha_1$. Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.5.1). Then, condition (iii), together with (3.9.2), yields, as far as $x(t)$ exists to the right of t_0,

$$D^+V(t, x)_{(3.5.1)} \leqslant g(t, V(t, x)) + M\| R(t, x)\|.$$

Define $\lambda(t) = M \| R(t, x(t))\|$, and choose $u_0 = V(t_0, x_0)$. An application of Theorem 1.4.1 shows that

$$V(t, x(t)) \leqslant r(t, t_0, u_0), \tag{3.9.3}$$

on the common interval of existence of $x(t)$ and $r(t, t_0, u_0)$, where $r(t, t_0, u_0)$ is the maximal solution of

$$u' = g(t, u) + \lambda(t), \qquad u(t_0) = u_0 .$$

Assume now that (I_1^*) holds. Then, given $\alpha_1 \geqslant 0$ and $t_0 \in J$, there exists a $\beta_1 = \beta_1(t_0, \alpha_1)$, which is continuous in t_0 for each α_1 and $\beta_1 \in \mathscr{K}$ for each t_0, such that, for every solution $u(t, t_0, u_0)$ of (3.9.1), the inequality

$$u(t, t_0, u_0) < \beta_1, \qquad t \geqslant t_0,$$

holds, whenever $u_0 \leqslant \alpha_1$ and, for every $T > 0$,

$$\int_{t_0}^{t_0+T} \phi(s)\, ds \leqslant \alpha_1 .$$

Since assumption (ii) holds, it is possible to choose a $\beta = \beta(t_0, \alpha)$ satisfying the relation

$$b(\beta) \geqslant \beta_1, \tag{3.9.4}$$

where β_1 is the function occurring in (I_1^*). Evidently, β is continuous in t_0 for each α and $\beta \in \mathscr{K}$ for each t_0. We claim that, with this β, definition (I_1) holds. If this is not true, there would exist a $t_1 > t_0$ such that

$$\| x(t_1, t_0, x_0)\| = \beta \tag{3.9.5}$$

and

$$\| x(t, t_0, x_0)\| \leqslant \beta, \qquad t \in [t_0, t_1].$$

For $t \in [t_0, t_1]$, take

$$\phi(t) = M \| R(t, x(t))\|.$$

Then we have

$$\begin{aligned} \int_{t_0}^{t_1} \phi(s)\, ds &= \int_{t_0}^{t_1} M \| R(s, x(s))\|\, ds \\ &\leqslant M \int_{t_0}^{t_1} \sup_{\|x\| \leqslant \beta} \| R(s, x)\| \\ &< M\alpha = \alpha_1 . \end{aligned}$$

We extend $\phi(t)$ continuously for all $t \geqslant t_0$ such that

$$\int_{t_0}^{\infty} \phi(s)\, ds < \alpha_1 .$$

To do this, it is enough to take $t_2 > t_1$, satisfying the inequality

$$t_2 - t_1 < \frac{2(\alpha_1 - \int_{t_0}^{t_1} \phi(s)\, ds)}{1 + \phi(t_1)},$$

to put $\phi(t_2) = 0$, and to take $\phi(t)$ linear on $[t_1, t_2]$ and $\phi(t) = 0$ for $t \geqslant t_2$.

Let $r^*(t, t_0, u_0)$ be the maximal solution of the perturbed differential equation (3.9.1) with $\phi(t)$ chosen as before. Because of (I_1^*), it would follow from $u_0 \leqslant \alpha_1$ and

$$\int_{t_0}^{t_0+T} \phi(s)\, ds < \alpha_1 ,$$

for every $T > 0$, that

$$r^*(t, t_0, u_0) < \beta_1 , \qquad t \geqslant t_0 .$$

But, on $[t_0, t_1]$, we have

$$r^*(t, t_0, u_0) \equiv r(t, t_0, u_0),$$

since $\lambda(t)$ and $\phi(t)$ are identical on this interval. Hence, $r(t_1, t_0, u_0) < \beta_1$. Thus, we get, from relations (3.9.5), (3.9.3), (3.9.4), and assumption (ii), the following absurdity:

$$b(\beta) \leqslant V(t_1, x(t_1)) \leqslant r(t_1, t_0, u_0) < \beta_1 \leqslant b(\beta).$$

This proves the integral stability of the trivial solution of (3.2.1).

COROLLARY 3.9.1. If the function $g(t, u) = \lambda(t)u$ enjoys the property that $\lambda \in C[J, R]$ and

$$\int_{t_0}^{\infty} \lambda(s)\, ds < \infty,$$

then it is a candidate in Theorem 3.9.1.

THEOREM 3.9.2. Under the assumptions of Theorem 3.9.1, the uniform integral stability of the solution $u = 0$ of (3.2.3) assures the uniform integral stability of the solution $x = 0$ of (3.2.1).

Proof. The proof is very much the same except to observe from (3.9.4) that β is independent of t_0 since β_1 does not depend on t_0 by assumption.

COROLLARY 3.9.2. The function $g(t, u) \equiv 0$ is admissible in Theorem 3.9.2.

THEOREM 3.9.3. Let the assumptions of Theorem 3.9.1 hold. Assume that the trivial solution of (3.2.3) is equi-asymptotically integrally stable. Then, the null solution of (3.2.1) is likewise equi-asymptotically integrally stable.

Proof. On the basis of Theorem 3.9.1, the trivial solution of (3.2.1) is equi-integrally stable. Let $\epsilon > 0$, $\alpha \geqslant 0$, and $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \alpha$. As in Theorem 3.9.1, we define $\alpha_1 = M\alpha$. Let $\beta = \beta(t_0, \alpha)$ be the same function obtained by relation (3.9.4), for which (I_1) holds.

Since (I_3^*) holds, it follows that, given $b(\epsilon) > 0$, $\alpha_1 \geqslant 0$, and $t_0 \in J$, there exists a pair of numbers $\gamma_1 = \gamma_1(t_0, \alpha_1, \epsilon)$ and $T = T(t_0, \alpha_1, \epsilon)$ such that, whichever be the function $\phi \in C[J, R_+]$ with

$$\int_0^\infty \phi(s)\, ds < \gamma_1, \tag{3.9.6}$$

every solution $u(t, t_0, u_0)$ of the perturbed scalar differential equation (3.9.1) satisfies

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 + T, \tag{3.9.7}$$

whenever $u_0 \leqslant \alpha_1$. We now choose a positive number $\gamma = \gamma(t_0, \alpha, \epsilon)$ so that

$$M\gamma = \gamma_1 \tag{3.9.8}$$

and maintain that, with the positive numbers T and γ so defined, (I_3) is satisfied. For otherwise, let $\{t_k\}$ be a sequence such that $t_k \geqslant t_0 + T$, $t_k \to \infty$ as $k \to \infty$. Suppose that there is a solution $x(t) = x(t, t_0, x_0)$ of the system (3.5.1) such that $\| x_0 \| \leqslant \alpha$ and $\| x(t_k, t_0, x_0)\| \geqslant \epsilon$. As in the proof of Theorem 3.9.1, condition (iii), in view of the fact that $V(t, x)$ is Lipschitzian, gives

$$D^+V(t, x(t)) \leqslant g(t, V(t, x(t))) + M\| R(t, x(t))\|. \tag{3.9.9}$$

If we now define $\phi(t) = M \| R(t, x(t))\|$, we have

$$\begin{aligned}\int_{t_0}^\infty \phi(s)\, ds &= \int_{t_0}^\infty M\| R(s, x(s))\|\, ds \\ &\leqslant M \int_{t_0}^\infty \sup_{\|x\| \leqslant \beta} \| R(s, x)\|\, ds \\ &< M\gamma = \gamma_1,\end{aligned}$$

using (3.9.8) and the fact that $\int_{t_0}^\infty \sup_{\|x\|\leqslant\beta} \| R(s, x)\|\, ds < \gamma$. This implies that, for solutions $u(t, t_0, u_0)$ of (3.9.1), (3.9.7) is true, because

of (3.9.6). Moreover, by (3.9.9) and the definition of $\phi(t)$, it follows from Theorem 1.4.1 that

$$V(t, x(t)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0, \tag{3.9.10}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.9.1). Hence, relations (3.9.10) and (3.9.7) and assumption (ii) of Theorem 3.9.1 lead us to the contradiction

$$b(\epsilon) \leqslant V(t_k, x(t_k)) \leqslant r(t_k, t_0, u_0) < b(\epsilon),$$

which proves the equi-asymptotic integral stability of the solution $x = 0$ of (3.2.1), and the theorem is complete.

THEOREM 3.9.4. Let the assumptions of Theorem 3.9.1 hold. Assume that the null solution of (3.2.3) is uniformly asymptotically integrally stable. Then the solution $x = 0$ of (3.2.1) is likewise uniformly asymptotically integrally stable.

Proof. We note that, in this case, the positive numbers T and γ_1 are independent of t_0, and therefore (3.9.8) implies that γ is also independent of t_0. The rest of the proof is just the same as that of Theorem 3.9.3.

If the function $g(t, u)$ is assumed to be nonincreasing in u for each $t \in J$, we can obtain integral stability notions from the stability notions of the trivial solution $u = 0$ of (3.2.3). To this end, we shall prove the following result.

THEOREM 3.9.5. Assume that the hypotheses of Theorem 3.9.1 hold. Let the function $g(t, u)$ be nonincreasing in u for each $t \in J$. Then, uniform asymptotic stability of the null solution $u = 0$ of (3.2.3) implies uniform asymptotic integral stability of the trivial solution of (3.2.1).

Proof. We first prove the uniform integral stability of the solution $x = 0$ of (3.2.1). By Theorem 3.3.5, uniform stability of the null solution of (3.2.3) implies the existence of a function $\beta_1 \in \mathscr{K}$ such that

$$u(t, t_0, u_0) \leqslant \beta_1(u_0), \qquad t \geqslant t_0. \tag{3.9.11}$$

Let now $\alpha \geqslant 0$ and $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \alpha$. Then we have, from (3.9.2), $V(t_0, x_0) \leqslant M\alpha$. Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.5.1) with $\| x_0 \| \leqslant \alpha$, and let

$$m(t) + \phi(t) = V(t, x(t)), \tag{3.9.12}$$

where

$$\phi(t) = M \int_t^t \| R(s, x(s))\| \, ds.$$

Using condition (iii) and inequality (3.9.2), we get

$$\begin{aligned} D^+m(t) &\leqslant D^+V(t, x(t))_{(3.5.1)} - M\| R(t, x(t))\| \\ &\leqslant D^+V(t, x(t))_{(3.2.1)} \\ &\leqslant g(t, V(t, x(t))), \end{aligned}$$

from which it follows, because of the monotonic nonincreasing character of $g(t, u)$ in u and the fact that $m(t) \geqslant V(t, x(t))$, that

$$D^+m(t) \leqslant g(t, m(t)).$$

By Theorem 1.4.1, we then have, as far as $x(t)$ exists to the right of t_0,

$$m(t) \leqslant r(t, t_0, u_0), \tag{3.9.13}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.2.3) with $u_0 = m(t_0)$. Let β be so chosen that

$$b(\beta) \geqslant \beta_1(M\alpha) + M\alpha. \tag{3.9.14}$$

This choice is clearly possible in view of the fact that $b(u) \to \infty$ as $u \to \infty$. It is evident that $\beta = \beta(\alpha)$ and that $\beta \in \mathscr{K}$. We claim that, with this β, the trivial solution $x = 0$ of (3.2.1) is uniformly integrally stable, whenever $\| x_0 \| \leqslant \alpha$ and, for every $T > 0$,

$$\int_{t_0}^{t_0+T} \sup_{\|x\| \leqslant \beta} \| R(s, x)\| \, ds < \alpha. \tag{3.9.15}$$

Assuming that this claim is false, there exists a $t_1 > t_0$ such that

$$\| x(t_1, t_0, x_0)\| = \beta \quad \text{and} \quad \| x(t, t_0, x_0)\| \leqslant \beta, \qquad t \in [t_0, t_1].$$

We are then led to the absurdity, because of relations (3.9.12), (3.9.14), (3.9.15), and assumption (ii),

$$\begin{aligned} b(\beta) \leqslant V(t_1, x(t_1)) &\leqslant r(t_1, t_0, u_0) + \phi(t_1) \\ &\leqslant \beta_1(M\alpha) + \int_{t_0}^{t_1} M\| R(s, x(s))\| \, ds \\ &\leqslant \beta_1(M\alpha) + M \int_{t_0}^{t_1} \sup_{\|x\| \leqslant \beta} \| R(s, x)\| \, ds \\ &< \beta_1(M\alpha) + M\alpha \\ &\leqslant b(\beta), \end{aligned}$$

thus proving (I_2).

To prove (I_4), we have, by uniform asymptotic stability of the solution $u = 0$ of (3.2.3) and Theorem 3.4.11, the inequality

$$u(t, t_0, u_0) \leqslant \beta_1(u_0)\sigma(t - t_0), \qquad t \geqslant t_0, \tag{3.9.16}$$

where $\beta_1 \in \mathscr{K}$ and $\sigma \in \mathscr{L}$. If we are now given $\epsilon > 0$, $\alpha \geqslant 0$, and $t_0 \in J$, we make the following choice: $\| x_0 \| \leqslant \alpha$, $M\gamma < b(\epsilon)$, and $\gamma = \min(\gamma, \alpha)$. Since, for any solution $x(t) = x(t, t_0, x_0)$ of (3.5.1), (3.9.13) is true, whenever $u_0 = V(t_0, x_0)$, relations (3.9.12) and (3.9.16), together with assumption (ii), given the inequality

$$\begin{aligned} b(\| x(t)\|) \leqslant V(t, x(t)) &\leqslant r(t, t_0, u_0) + M \int_{t_0}^{t} \| R(s, x(s))\| \, ds, \qquad t \geqslant t_0 \\ &\leqslant \beta_1(M\alpha)\sigma(t - t_0) + M \int_{t_0}^{t} \sup_{\|x\| \leqslant \beta} \| R(s, x)\| \, ds \\ &\leqslant \beta_1(M\alpha)\sigma(t - t_0) + M\gamma. \end{aligned}$$

Since $\sigma \in \mathscr{L}$, there exists a $T = T(\alpha, \epsilon)$ such that

$$\sigma(t - t_0) < \frac{b(\epsilon) - M\gamma}{\beta_1(M\alpha)}, \qquad t \geqslant t_0 + T,$$

and hence, for $t \geqslant t_0 + T$, we would have

$$b(\| x(t)\|) < b(\epsilon),$$

which implies that

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0 + T,$$

provided $\| x_0 \| \leqslant \alpha$ and (3.9.15) is satisfied. The theorem is proved.

COROLLARY 3.9.3. Assume that there exists a function $V(t, x)$ satisfying the following properties:

(i) $V \in C[J \times R^n, R_+]$, $V(t, 0) = 0$, $t \in J$, and $V(t, x)$ is positive definite and Lipschitzian in x for a constant $M > 0$.

(ii) $D^+V(t, x)_{(3.2.1)} \leqslant -C(\| x \|)$, $(t, x) \in J \times R^n$, where the function $C \in \mathscr{K}$.

Then, the trivial solution of (3.2.1) is uniformly asymptotically integrally stable.

Proof. By relation (3.9.2), we have $V(t, x) \leqslant M \| x \|$. This fact, together with condition (ii), is sufficient to arrive at the differential inequality

$$D^+V(t, x)_{(3.2.1)} \leqslant g(t, V(t, x)),$$

where $g(t, V) = -C(V(t, x)/M)$. It is now immediate, by Corollary 3.4.2, that the null solution of (3.2.3) is uniformly asymptotically stable. Moreover, $g(t, u)$ defined previously has all the properties required by Theorem 3.9.5. Hence, the corollary is a consequence of Theorem 3.9.5.

3.10. L^p-stability

DEFINITION 3.10.1. The trivial solution $x = 0$ of (3.2.1) is said to be

(L_1) *equi-L^p stable* if (S_1) holds and there exists a $\delta_0 = \delta_0(t_0) > 0$ such that the inequality $\| x_0 \| \leqslant \delta_0$ implies

$$\int_{t_0}^{\infty} \| x(s, t_0, x_0)\|^p \, ds < \infty; \tag{3.10.1}$$

(L_2) *uniform-L^p stable* if (S_2) holds, the δ_0 in (L_1) is independent of t_0, and the integral (3.10.1) converges uniformly in t_0.

The following example shows that L^p stability need not necessarily imply asymptotic stability, and vice versa. Thus, they are different concepts.

Consider the linear equation

$$x' = \frac{g'(t)}{g(t)} x, \tag{3.10.2}$$

whose general solution

$$x(t, t_0, x_0) = \frac{g(t)}{g(t_0)} x_0 .$$

If $g(t)$ is a continuously differentiable, bounded L^1 function on J which does not tend to zero as $t \to \infty$, then the trivial solution of (3.10.2) is L^1 stable but not asymptotically stable. On the other hand, if $g(t) = [\log(t + 2)]^{-1}$, then the trivial solution of (3.10.2) is asymptotically stable but not L^p stable for any $p > 0$.

Analogous to the notions (L_1) and (L_2), we require the L^1 stability notions for the null solution of the scalar differential equation (3.2.3).

DEFINITION 3.10.2. The null solution $u = 0$ of (3.2.3) is (L_1^*) *equi-L^1 stable* if (S_1^*) holds and there exists a $\delta_0 = \delta_0(t_0)$ such that $u_0 \leqslant \delta_0$ implies

$$\int_{t_0}^{\infty} u(s, t_0, u_0) \, ds < \infty. \tag{3.10.3}$$

The definition of uniform L^1 stability is clear.

THEOREM 3.10.1. Assume that there exist functions $V(t, x)$ and $g(t, u)$ satisfying the following properties:

(i) $g \in C[J \times R_+, R]$ and $g(t, 0) = 0, t \in J$.

(ii) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) = 0$, $t \in J$, $V(t, x)$ is locally Lipschitzian in x, and

$$A\|x\|^p \leqslant V(t, x), \qquad (t, x) \in J \times S_\rho . \tag{3.10.4}$$

(iii) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times S_\rho$.

Then, the equi-L^1 stability of the null solution of (3.2.3) implies the equi-L^p stability of the null solution $x = 0$ of (3.2.1).

Proof. Assume that the null solution of (3.2.3) is equi-L^1 stable. Then, it is equistable and there exists a $\delta_0 = \delta_0(t_0)$ such that $u_0 \leqslant \delta_0$ implies (3.10.3). By Theorem 3.3.1, the equistability of the null solution of (3.2.1) follows, and therefore, to prove (L_1), it remains to show that there exists a $\delta_0 = \delta_0(t_0)$ such that $\|x_0\| \leqslant \delta_0$ assures (3.10.1). Since $V(t, x)$ is continuous and $V(t, 0) = 0, t \in J$, there exists a positive number $\delta_0 = \delta_0(t_0)$ satisfying the inequalities

$$\|x_0\| \leqslant \delta_0 , \qquad V(t_0 , x_0) \leqslant \delta_0$$

together. As in the proof of Theorem 3.1.1, using condition (iii) and Theorem 3.1.1, we get, by choosing $u_0 = V(t_0 , x_0)$, the inequality

$$V(t, x(t, t_0 , x_0)) \leqslant r(t, t_0 , u_0), \qquad t \geqslant t_0 ,$$

where $x(t, t_0 , x_0)$ is any solution of (3.2.1) such that $\|x_0\| \leqslant \delta_0$ and $r(t, t_0 , u_0)$ is the maximal solution of (3.2.3). From this, the desired result (3.10.1) follows, using assumptions (3.10.4) and (3.10.3). The proof is thus complete.

THEOREM 3.10.2. Assume that there exists a function $V(t, x)$ with the following properties:

(i) $V \in C[J \times S_\rho , R_+]$, $V(t, 0) = 0, t \in J$, and $V(t, x)$ is positive definite and locally Lipschitzian in x.

(ii) $D^+V(t, x) \leqslant -C\|x\|^p$, $(t, x) \in J \times S_\rho$, $C > 0$.

Then the null solution $x = 0$ of (3.2.1) is equi-L^p stable.

Proof. By Corollary 3.3.2, it follows that (S_1) holds. Let $t_0 \in J$, and let $\delta_0 = \delta_0(t_0) > 0$ be such that, if $\|x_0\| \leqslant \delta_0$, then $(t, x(t, t_0, x_0)) \in J \times S_\rho$ for $t \geqslant t_0$. This is possible because (S_1) is valid. Define

$$m(t) = V(t, x(t, t_0 , x_0)) + C \int_{t_0}^{t} \|x(s, t_0 , x_0)\|^p \, ds.$$

Then, condition (ii), in view of Theorem 3.1.3, gives the inequality

$$m(t) \leqslant m(t_0), \qquad t \geqslant t_0,$$

from which there further results

$$\int_{t_0}^{\infty} \| x(s, t_0, x_0)\|^p \, ds \leqslant \frac{1}{c} V(t_0, x_0).$$

It is clear that the null solution $x = 0$ is equi-L^p stable.

THEOREM 3.10.3. Let the assumptions of Theorem 3.10.1 hold. Suppose further that

$$V(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times S_\rho, \qquad a \in \mathscr{K}.$$

Then, the uniform-L^1 stability of the solution $u = 0$ of (3.2.3) implies the uniform-L^p stability of the solution $x = 0$ of (3.2.1).

Proof. By the assumption, the null solution of (3.2.3) is uniformly stable, and there exists a δ_0 independent of t_0 such that (3.10.3) holds uniformly in t_0 whenever $u_0 \leqslant \delta_0$. Consequently, the uniform stability of the solution $x = 0$ of (3.2.1) follows from Theorem 3.3.4. To prove that (L_2) holds, we follow the proof of Theorem 3.10.1 and we choose $u_0 = a(\| x_0 \|)$, thereby deducing $\delta_0 = a^{-1}(\delta_0)$. It is evident that δ_0 is independent of t_0 and the integral (3.10.1) is uniformly convergent in t_0. This proves the theorem.

COROLLARY 3.10.1. Assume that there exists a function $V(t, x)$ verifying the following conditions:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is locally Lipschitzian, and, for $(t, x) \in J \times S_\rho$,

$$A\| x \|^p \leqslant V(t, x) \leqslant B\| x \|^p, \qquad A, B > 0. \tag{3.10.5}$$

(ii) $D^+V(t, x) \leqslant -C \| x \|^p$, $C > 0$, $(t, x) \in J \times S_\rho$.

Then, the null solution of (3.2.1) is uniform-L^p stable.

Proof. Assumption (ii), in virtue of (3.10.5), reduces to

$$D^+V(t, x) \leqslant g(t, V(t, x)),$$

where $g(t, u) = -Cu/B$, and hence it is easy to check that the solution $u = 0$ of (3.2.1) is uniform-L^1 stable. Now, the assertion of the corollary is a consequence of Theorem 3.10.3.

Although L^p stability and asymptotic stability are different concepts, under certain conditions L^p stability implies asymptotic stability, as shown in the following:

THEOREM 3.10.4. Let the trivial solution of (3.2.1) be L^p stable, and let there exist a constant $M > 0$ such that

$$\|f(t, x)\| \leqslant M, \qquad (t, x) \in J \times S_\rho .$$

Then, the trivial solution $x = 0$ of (3.2.1) is asymptotically stable.

Proof. Assume that there is a solution $x(t, t_0, x_0)$ of (3.2.1) such that $\| x_0 \| \leqslant \delta_0(t_0)$ and $\lim_{t\to\infty} x(t, t_0, x_0) \neq 0$. Then, there exists an $\epsilon > 0$ and a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that

$$\| x(t_k, t_0, x_0)\| \geqslant 2\epsilon \qquad \text{for all} \quad k.$$

By assumption,

$$\| x'(t, t_0, x_0)\| = \|f(t, x(t, t_0, x_0))\| \leqslant M, \qquad t \geqslant t_0 ,$$

and hence there is a number $\lambda > 0$ satisfying

$$\| x(t, t_0, x_0)\| \geqslant \epsilon, \qquad t_k \leqslant t \leqslant t_k + \lambda$$

for each k. This contradicts (3.10.1), and the theorem is proved.

We shall next consider some converse theorems for L^p stability.

THEOREM 3.10.5. Let us assume that

(i) the function $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $f_x(t, x)$ exists and is continuous for $(t, x) \in J \times S_\rho$;

(ii) the solution $x(t, 0, x_0)$ of (3.2.1) satisfies

$$\| x(t, 0, x_0)\| \leqslant \beta\| x_0 \|, \qquad t \geqslant 0, \beta > 0, \tag{3.10.6}$$

and $\int_0^\infty \| x(s, 0, x_0)\|^p \, ds < \infty$;

(iii) the function $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and $g_u(t, u)$ exists and is continuous for $(t, u) \in J \times R_+$;

(iv) the solution $u(t, 0, u_0)$ of (3.2.3) verifies the estimate

$$\gamma u_0 \leqslant \int_0^\infty u(s, 0, u_0)\, ds, \qquad \gamma > 0. \tag{3.10.7}$$

Then, there exists a function $V(t, x)$ with the following properties:

(1) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ is continuously differentiable, and

$$A\| x \|^p \leqslant V(t, x), \qquad (t, x) \in J \times S_\rho, \ A > 0;$$

(2) $V'(t, x) = g(t, V(t, x)), (t, x) \in J \times S_\rho$.

Proof. By assumptions (i) and (iii), the continuity and differentiability with respect to the initial values of the solutions $x(t, t_0, x_0)$, $u(t, t_0, u_0)$ of (3.2.1), (3.2.3), respectively, follows as in Theorem 3.6.4. Moreover, denoting $x(t, 0, x_0)$ by x, we see that $x_0 = x(0, t, x)$, because of uniqueness of solutions.

We choose a continuous function $\mu(x)$ such that $\mu(0) = 0$, $\partial\mu(x)/\partial x$ exists and is continuous, and

$$\alpha\| x \|^p \leqslant \mu(x), \qquad \alpha > 0, \qquad x \in S_\rho. \tag{3.10.8}$$

We then define the function

$$V(t, x) = u(t, 0, \mu(x(0, t, x))) + \int_0^\infty u(s, 0, \mu(x(0, t, x)))\, ds.$$

It is clear that $V \in C[J \times S_\rho, R_+]$ and $V(t, 0) \equiv 0$, since $x(0, t, 0) \equiv 0$ and $u(t, 0, 0) \equiv 0$. Furthermore, taking into account the fact that the solutions $x(t, t_0, x_0)$ and $u(t, t_0, u_0)$ are differentiable with respect to their arguments, we have

$$\begin{aligned}
V'(t, x) &= u'(t, 0, \mu(x(0, t, x))) \\
&\quad + \frac{\partial u}{\partial u_0} \cdot \frac{\partial \mu}{\partial x}\left[\frac{\partial x(0, t, x)}{\partial t_0} + \frac{\partial x(0, t, x)}{\partial x_0} \cdot f(t, x)\right] \\
&\quad + \int_0^\infty \frac{\partial u}{\partial u_0} \frac{\partial u}{\partial x}\left[\frac{\partial x(0, t, x)}{\partial t_0} + \frac{\partial x(0, t, x)}{\partial x_0} \cdot f(t, x)\right] ds \\
&= g(t, V(t, x)),
\end{aligned}$$

because of (3.6.11).

Using the relations (3.10.6), (3.10.7), (3.10.8), and the definition of $V(t, x)$, we obtain

$$\begin{aligned}
V(t, x) &\geqslant \int_0^\infty u(s, 0, \mu(x(0, t, x)))\, ds \\
&\geqslant \gamma\mu(x(0, t, x)) \\
&\geqslant \gamma\alpha\| x(0, t, x)\|^p \\
&\geqslant \frac{\gamma\alpha}{\beta^p}\| x \|^p \equiv A\| x \|^p.
\end{aligned}$$

The proof is complete.

We notice that the full force of assumption (ii), that is, L^p-nature of solutions $x(t, 0, x_0)$, is not used in the proof of Theorem 3.10.5. However, the linear character of the estimate in (3.10.6) is crucial in the proof. We give below a different type of converse theorem.

THEOREM 3.10.6. Let assumption (i) of Theorem 3.10.5 hold. Furthermore, suppose that

$$\beta_2(\| x_0 \|)\lambda(t) \leqslant \| x(t, t_0, x_0)\| \leqslant \beta_1(\| x_0 \|)\lambda(t), \qquad t \geqslant t_0, \tag{3.10.9}$$

where $\beta_1, \beta_2 \in \mathscr{K}$ and $\lambda \in C[J, R_+]$ such that $\lambda \in L^p$. Then, there exists a function $V(t, x)$ satisfying the following:

(1) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ possesses continuous partial derivatives with respect to t and the components of x, and $V(t, x)$ is positive definite and decrescent.

(2) $V'(t, x) = -\| x \|^p$, $(t, x) \in J \times S_\rho$.

Proof. We define the function

$$V(t, x) = \int_t^\infty \| x(s, t, x)\|^p \, ds + \int_0^\infty \| x(s, t, x)\|^p \, ds.$$

Then, it is evident that $V \in C[J \times S_\rho, R_+]$, on the basis of assumption (i). Moreover, the differentiability of the solution $x(t, t_0, x_0)$ with respect to the initial values assures that $V(t, x)$ is continuously differentiable. Hence,

$$\begin{aligned} V'(t, x) &= -\| x(t, t, x)\|^p \\ &\quad + \int_t^\infty p\| x(s, t, x)\|^{p-2} x(s, t, x) \left[\frac{\partial x(s, t, x)}{\partial t_0} + \frac{\partial x(s, t, x)}{\partial x_0} \cdot f(t, x)\right] ds \\ &\quad + \int_0^\infty p\| x(s, t, x)\|^{p-2} x(s, t, x) \left[\frac{\partial x(s, t, x)}{\partial t_0} + \frac{\partial x(s, t, x)}{\partial x_0} \cdot f(t, x)\right] ds \\ &= -\| x \|^p, \end{aligned}$$

on account of the relation (3.6.11).

Also, using the fact that $\lambda \in L^p$ and the estimate (3.10.9), we have, successively,

$$\begin{aligned} V(t, x) &\geqslant \int_0^\infty \| x(s, t, x)\|^p \, ds \\ &\geqslant \beta_2{}^p(\| x \|) \int_0^\infty \lambda^p(s) \, ds \\ &= b(\| x \|), \qquad b \in \mathscr{K}, \end{aligned}$$

and

$$\begin{aligned} V(t, x) &\leqslant 2 \int_0^\infty \| x(s, t, x)\|^p \, ds \\ &\leqslant 2\beta_1{}^p(\| x \|) \int_0^\infty \lambda^p(s) \, ds \\ &\equiv a(\| x \|), \qquad a \in \mathscr{K}. \end{aligned}$$

The proof of the theorem is complete.

3.11. Partial stability

Let us consider a differential system of the form

$$\left.\begin{aligned} x' &= G(t, x, y), \qquad x(t_0) = x_0, \\ y' &= H(t, x, y), \qquad y(t_0) = y_0, \end{aligned}\right\} \tag{3.11.1}$$

where $G \in C[J \times S_\rho \times R^m, R^n]$, $H \in C[J \times S_\rho \times R^m, R^m]$, and $G(t, 0, 0) = 0$, $H(t, 0, 0) = 0$, $t \in J$. Let us denote a solution of (3.11.1) by $x(t) = x(t, t_0, x_0, y_0)$, $y(t) = y(t, t_0, x_0, y_0)$.

DEFINITION 3.11.1. The trivial solution $x = 0$, $y = 0$ of (3.11.1) is said to be (P_1) *partially equistable* with respect to components x if, for each $\epsilon > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ which is continuous in t_0 for each ϵ such that the inequality

$$\| x_0 \| + \| y_0 \| \leqslant \delta$$

implies

$$\| x(t, t_0, x_0, y_0)\| < \epsilon, \qquad t \geqslant t_0 .$$

Corresponding to the group of definitions (S_1)–(S_{10}), we may formulate (P_1)–(P_{10}).

THEOREM 3.11.1. Assume that there exist functions $V(t, x, y)$ and $g(t, u)$ satisfying the following properties:

(i) $V \in C[J \times S_\rho \times R^m, R_+]$, $V(t, 0, 0) \equiv 0$, and $V(t, x, y)$ is locally Lipschitzian in x and y.

(ii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and

$$\begin{aligned} D^+V(t, x, y) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hG(t, x, y), y + hH(t, x, y)) \\ &\quad - V(t, x, y)] \\ &\leqslant g(t, V(t, x, y)), \qquad (t, x, y) \in J \times S \ \times R^m . \end{aligned}$$

Then

(1) if the solution $u = 0$ of (3.2.3) is equistable and

$$b(\| x \|) \leqslant V(t, x, y), \qquad b \in \mathscr{K}, \tag{3.11.2}$$

the trivial solution of (3.11.1) is partially equistable;

(2) if the solution $u = 0$ of (3.2.3) is uniformly stable and

$$b(\| x \|) \leqslant V(t, x, y) \leqslant a(\| x \| + \| y \|), \qquad a, b \in \mathscr{K}, \tag{3.11.3}$$

the trivial solution of (3.11.1) is partially uniformly stable;

(3) if the solution $u = 0$ of (3.2.3) is equi-asymptotically stable and (3.11.2) holds, the trivial solution of (3.11.1) is partially equi-asymptotically stable;

(4) if the solution $u = 0$ of (3.2.3) is uniformly asymptotically stable and (3.11.3) holds, the trivial solution of (3.11.1) is partially uniformly asymptotically stable.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in J$. Assume that the solution $u = 0$ of (3.2.3) is equistable. Then, given $b(\epsilon) > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that, whenever $u_0 \leqslant \delta$, we have

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 . \tag{3.11.4}$$

Choose $u_0 = V(t_0, x_0, y_0)$. Since $V(t, 0, 0) \equiv 0$ and $V(t, x, y)$ is continuous, there exists a $\delta_1 = \delta_1(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that the inequalities

$$\| x_0 \| + \| y_0 \| \leqslant \delta_1 , \qquad V(t_0, x_0, y_0) \leqslant \delta \tag{3.11.5}$$

hold at the same time. We maintain that, with this δ_1, (P_1) is satisfied. For otherwise, suppose that there exists a $t_1 > t_0$ for which

$$\| x(t_1) \| = \epsilon, \qquad \| x(t) \| \leqslant \epsilon, \qquad t \in [t_0, t_1],$$

whenever $\| x_0 \| + \| y_0 \| \leqslant \delta_1$, so that

$$b(\epsilon) \leqslant V(t_1, x(t_1), y(t_1)). \tag{3.11.6}$$

This implies that $\| x(t) \| < \rho$ for $t \in [t_0, t_1]$, and hence condition (ii), together with the choice $u_0 = V(t_0, x_0, y_0)$, yields, on the basis of Theorem 3.1.1,

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \qquad t \in [t_0, t_1], \tag{3.11.7}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.2.3). Relations (3.11.4), (3.11.6), and (3.11.7) lead to the absurdity

$$b(\epsilon) \leqslant V(t_1, x(t_1), y(t_1)) \leqslant r(t_1, t_0, u_0) < b(\epsilon),$$

proving the validity of (P_1). This establishes (1).

To prove the statement concerning (2), we have to choose $u_0 = a(\| x_0 \| + \| y_0 \|)$ so that δ_1 may be taken equal to $a^{-1}(\delta)$. Evidently, δ_1 is independent of t_0, and, as a result, (P_2) is satisfied.

Let us assume (S_5^*), so that (S_1^*) and (S_3^*) hold. Then, given $b(\epsilon) > 0$, $t_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ satisfying

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 + T,$$

provided $u_0 \leqslant \delta_0$. As previously, choosing $u_0 = V(t_0, x_0, y_0)$, we can find a $\mathring{\delta}_0 = \delta_0(t_0)$ obeying the inequalities

$$\| x_0 \| + \| y_0 \| \leqslant \mathring{\delta}_0, \qquad V(t_0, x_0, y_0) \leqslant \delta_0$$

simultaneously. It is easy to see that (P_1) is true, which implies that the inequality (3.11.7) is valid for all $t \geqslant t_0$. If we now suppose that there exists a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$, and $t_k \to \infty$ as $k \to \infty$, such that $\| x(t_k)\| \geqslant \epsilon$ for some solution $x(t), y(t)$ of (3.11.1) with the property that $\| x_0 \| + \| y_0 \| \leqslant \mathring{\delta}_0$, we are encountered with the following contradiction:

$$b(\epsilon) \leqslant V(t_k, x(t_k), y(t_k)) \leqslant r(t_k, t_0, u_0) < b(\epsilon).$$

Thus, (P_3) is true, which in turn shows the partial equi-asymptotic stability of the trivial solution of (3.11.1).

Finally, analogous to the proof of (2), it is easy to verify the assertion occuring in (4). This completes the proof of the theorem.

THEOREM 3.11.2. Suppose that the trivial solution of (3.11.1) is partially uniformly stable with respect to components x. Then, it is uniformly stable if the following conditions hold:

(i) $H(t, x, y)$ satisfies a Lipschitz condition in x and y for a constant $K > 0$.

(ii) The trivial solution of

$$y' = H(t, 0, y) \tag{3.11.8}$$

is uniformly asymptotically stable.

Proof. Consider the system

$$y' = H(t, 0, y) + [H(t, x, y) - H(t, 0, y)]. \tag{3.11.9}$$

By assumption (i), we have

$$\| H(t, x, y) - H(t, 0, y)\| \leqslant K\| x \|. \tag{3.11.10}$$

Treating (3.11.9) as a perturbed system of (3.11.8), hypotheses (i) and (ii) assure, on the basis of Theorem 3.8.1, that the solution $y = 0$ of (3.11.8) is T_1-totally stable. It therefore follows that there exist positive numbers $\delta_1(\epsilon)$, $\delta_2(\epsilon)$ such that every solution $y(t)$ of (3.11.9) verifies the inequality

$$\| y(t, t_0, x_0, y_0)\| < \epsilon, \qquad t \geqslant t_0,$$

provided that $\| y_0 \| < \delta_1(\epsilon)$ and

$$\| H(t, x, y) - H(t, 0, y)\| < \delta_2(\epsilon), \qquad \| y \| < \epsilon, \qquad t \in J. \tag{3.11.11}$$

If $\delta_3(\epsilon) = \delta_2(\epsilon)/K$, we infer, from (3.11.10), that relation (3.11.11) is valid whenever $\| x \| < \delta_3(\epsilon)$. Since (P_2) is assumed to hold for the system (3.11.1), it follows that, given $\delta_3(\epsilon) > 0$, $t_0 \in J$, there exists a $\delta_4(\epsilon) > 0$ such that $\| x_0 \| + \| y_0 \| < \delta_4(\epsilon)$ implies

$$\| x(t, t_0, x_0, y_0)\| < \delta_3(\epsilon), \qquad t \geqslant t_0 .$$

Choose $\delta(\epsilon) = \min[\delta_1(\epsilon), \delta_4(\epsilon)]$. Then, $\| x_0 \| + \| y_0 \| < \delta(\epsilon)$ guarantees (3.11.11), and

$$\| x(t, t_0, x_0, y_0)\| < \delta_3(\epsilon) < \epsilon, \qquad t \geqslant t_0,$$

so that

$$\| y(t, t_0, x_0, y_0)\| < \epsilon, \qquad t \geqslant t_0 .$$

Thus, the uniform stability of the trivial solution of (3.11.1) is proved.

THEOREM 3.11.3. If conditions (i) and (ii) of Theorem 3.11.2 hold, the partial uniform asymptotic stability of the trivial solution of (3.11.1) assures the uniform asymptotic stability of the same trivial solution.

Proof. Since the trivial solution of (3.11.1) is partially uniformly asymptotically stable, we may write

$$\| x(t, t_0, x_0, y_0)\| \leqslant \beta(\| x_0 \| + \| y_0 \|)\sigma(t - t_0) \tag{3.11.12}$$

for $t \geqslant t_0$, $\beta \in \mathscr{K}$, and $\sigma \in \mathscr{L}$. Let

$$R(t, y) = H(t, x(t), y) - H(t, 0, y).$$

Then, because of (3.11.10) and (3.11.12), we deduce that

$$\| R(t, y)\| \leqslant K\beta(\| x_0 \| + \| y_0 \|)\sigma(t - t_0). \tag{3.11.13}$$

Consider the perturbed system

$$y' = H(t, 0, y) + R(t, y), \tag{3.11.14}$$

and let $V(t, y)$ be the Lyapunov function constructed according to Theorem 3.6.9. If $\| y_0 \| < \delta(\delta_0) = \rho_0$, it follows that

$$D^+V(t, y)_{(3.11.14)} \leqslant -C[V(t, y)] + M\| R(t, y)\|, \tag{3.11.15}$$

where $C \in \mathscr{K}$. Let $\| x_0 \| + \| y_0 \| < \delta_0$, where δ_0 is the number occuring in the definition of partial uniform asymptotic stability. It is easy to deduce, taking into account relations (3.11.13) and (3.11.15), that

$$D^+V(t, y)_{(3.11.14)} \leqslant g(t, V(t, y)),$$

where

$$g(t, u) = -C(u) + MK\beta(\delta_0)\sigma(t).$$

Notice that $R(t, y)$ satisfies a Lipschitz condition in y because of condition (i). Let $0 < \alpha \leqslant \beta < \rho_0$ be given, and let $K_1(\alpha, \beta) = \frac{1}{2}C(\alpha)$. The fact that $\sigma \in \mathscr{L}$ shows that there exists a $\theta(\alpha, \beta) \geqslant 0$ such that

$$\sigma(t) \leqslant \frac{C(\alpha)}{2MK\beta(\delta_0)} \qquad \text{if} \quad t \geqslant \theta(\alpha, \beta).$$

Consequently, if $\alpha \leqslant u \leqslant \beta$, $t \geqslant \theta(\alpha, \beta)$, we have

$$\begin{aligned} g(t, u) &= -C(u) + MK\beta(\delta_0)\sigma(t) \\ &\leqslant -C(\alpha) + \tfrac{1}{2}C(\alpha) = -K_1(\alpha, \beta). \end{aligned}$$

The hypotheses of Theorem 3.4.10 being verified, the conclusion follows as an application of Theorem 3.8.3.

3.12. Stability of differential inequalities

In this section, we shall be concerned with the differential inequality of the form

$$\| x' - f(t, x)\| \leqslant g_1(t, \| x \|), \tag{3.12.1}$$

which holds for $\| x \| < \rho$, where $f \in C[J \times R^n, R^n]$, $f(t, 0) \equiv 0$, and $g_1 \in C[J \times R_+, R_+]$.

Definition 3.12.1. Let $x(t)$ be a function defined and continuous for $t \geqslant t_0 \geqslant 0$. Suppose that $x(t)$ has the derivative $x'(t)$ and it satisfies (3.12.1) for $t \in [t_0, \infty) - S$, where S is an atmost countable set of $[t_0, \infty)$. Then $x(t)$ is said to be a *solution* of the differential inequality (3.12.1).

If $g_1(t, u) \equiv 0$, it is understood that S is empty and $x(t)$ is a solution of the differential equation

$$x' = f(t, x), \qquad x(t_0) = x_0 . \tag{3.12.2}$$

We wish to consider the stability properties of the differential inequality (3.12.1) with respect to origin.

Theorem 3.12.1. Let the following conditions hold:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$,

$$b(\| x \|) \leqslant V(t, x), \qquad b \in \mathscr{K}, \qquad (t, x) \in J \times S_\rho , \tag{3.12.3}$$

and

$$| V(t, x) - V(t, y)| \leqslant L\| x - y \|, \qquad (t, x), (t, y) \in J \times S_\rho , \tag{3.12.4}$$

L being a constant.

(ii) $g_2 \in C[J \times R_+, R_+]$, $g_2(t, 0) \equiv 0$, and

$$D^+V(t, x)_{(3.12.2)} \leqslant g_2(t, V(t, x)), \qquad (t, x) \in J \times S_\rho ; \tag{3.12.5}$$

(iii) $g_1(t, 0) \equiv 0$, and $g_1(t, u)$ is nondecreasing in u for $t \in J$.

Then, the stability properties of the trivial solution of (3.2.3) with

$$g(t, u) = Lg_1(t, b^{-1}(u)) + g_2(t, u) \tag{3.12.6}$$

imply the same kind of stability properties of the differential inequality (3.12.1) with respect to origin.

Proof. Assume that (S_1^*) holds. We shall only prove the corresponding conclusion and omit the rest.

Let $x(t)$ be any solution of (3.12.1) such that $V(t_0, x(t_0)) \leqslant u_0$. Defining $m(t) = V(t, x(t))$, we see, for small $h > 0$, that

$$\begin{aligned} m(t + h) - m(t) \leqslant\ & L\| x(t + h) - x(t) - hf(t, x(t))\| \\ & + V(t + h, x(t) + hf(t, x(t))) - V(t, x(t)) \end{aligned}$$

because of (3.12.4). Using (3.12.1), (3.12.3), (3.12.5), (3.12.6), and the monotonic character of $g_1(t, u)$ in u, we obtain the inequality

$$D^+m(t) \leqslant g(t, m(t)).$$

By Theorem 1.4.1, it follows that

$$m(t) = V(t, x(t)) \leqslant r(t, t_0, u_0), \tag{3.12.7}$$

for those values of $t \geqslant t_0$ for which $\| x(t)\| < \rho$, $r(t, t_0, u_0)$ being the maximal solution of (3.2.3).

Let now $\epsilon > 0$, $t_0 \in J$ be given. If $\| x \| = \epsilon$, it follows from (3.12.3) that

$$b(\epsilon) \leqslant V(t, x). \tag{3.12.8}$$

Since (S_1^*) holds, given $b(\epsilon) > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that $u_0 \leqslant \delta$ implies

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0. \tag{3.12.9}$$

Choose $u_0 = L \| x(t_0)\|$ and $\hat{\delta} = \hat{\delta}(t_0, \epsilon) = \delta/L$. Suppose now that a solution $x(t)$ of (3.12.1) such that $\| x(t_0)\| \leqslant \hat{\delta}$ has the property that $\| x(t_1)\| = \epsilon$ and $\| x(t)\| \leqslant \epsilon < \rho$ for $t \in [t_0, t_1]$, $t_1 > t_0$. This would mean, in view of relations (3.12.7), (3.12.8), and (3.12.9),

$$b(\epsilon) \leqslant V(t_1, x(t_1)) \leqslant r(t_1, t_0, u_0) < b(\epsilon).$$

This contradiction proves that $\| x(t)\| < \epsilon$, $t \geqslant t_0$, whenever $\| x(t_0) \leqslant \hat{\delta}$, and the proof is complete.

REMARK 3.12.1. Theorem 3.12.1 includes many special cases. If $g_1(t, u) \equiv 0$, we obtain the stability theorems for the differential system (3.12.2), whereas, if $\| R(t, x)\| \leqslant g_1(t, \| x \|)$ for $\| x \| < \rho$, $R \in C[J \times R^n, R^n]$, we deduce the stability properties of the trivial solution of (3.12.2) with respect to permanent perturbations $R(t, x)$.

THEOREM 3.12.2. Assume that conditions (i), (ii), and (iii) of Theorem 3.12.1 hold. Furthermore, suppose that the solutions $u(t, t_0, u_0)$ of (3.2.3) with $g(t, u)$ given by (3.12.6) for $0 \leqslant u_0 \leqslant \alpha$ have the property that $\lim_{t\to\infty} u(t, t_0, u_0) = 0$. Then, every solution $x(t)$ of (3.12.1) starting in the set

$$\Omega = [x \in R^n : V(t, x) \leqslant \alpha, t \geqslant 0]$$

tends to zero as $t \to \infty$. In other words, the set Ω is the domain of attraction.

Proof. Let $x(t)$ be any solution of (3.12.1) such that $x(t_0) \in \Omega$. Consider the function $m(t) = V(t, x(t))$. It is easy to obtain, as before, the differential inequality

$$D^+m(t) \leqslant g(t, m(t)),$$

and, consequently, the estimate

$$V(t, x(t)) = m(t) \leqslant r(t, t_0, \alpha), \qquad t \geqslant t_0,$$

where $r(t, t_0, \alpha)$ is the maximal solution of (3.12.6), with $u_0 = \alpha$. The assumption that $r(t, t_0, \alpha) \to 0$ as $t \to \infty$ and $V(t, x)$ is positive definite now assures that Ω is the domain of attraction.

Notice that this theorem brings out an important feature of comparison principle which is overlooked at times, that is, the behavior of the particular solution $u(t, t_0, u_0)$, with $V(t_0, x_0) \leqslant u_0$ determines for $t \geqslant t_0$, not just the behavior of the particular solution of (3.12.1) with $x(t_0) = x_0$, but, indeed, of all the solutions of (3.12.1) for which $V(t_0, x_0) \leqslant u_0$.

3.13. Boundedness and Lagrange stability

We consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{3.13.1}$$

where $f \in C[J \times R^n, R^n]$. We shall assume, for convenience, that f is smooth enough to ensure global existence of solutions of (3.13.1). We shall not require that $f(t, 0) \equiv 0$. To the different types of stability, there correspond different types of boundedness. Some important types are defined in the following:

DEFINITION 3.13.1. The differential system (3.13.1) is said to be

(B_1) *equibounded* if, for each $\alpha \geqslant 0$, $t_0 \in J$, there exists a positive function $\beta = \beta(t_0, \alpha)$, which is continuous in t_0 for each α, such that the inequality

$$\| x_0 \| \leqslant \alpha$$

implies

$$\| x(t, t_0, x_0)\| < \beta, \qquad t \geqslant t_0;$$

(B_2) *uniform bounded* if the β in (B_1) is independent of t_0;

(B_3) *quasi-equi-ultimately bounded* if, for each $\alpha \geqslant 0$ and $t_0 \in J$, there exist positive numbers N and $T = T(t_0, \alpha)$ such that the inequality

$$\| x_0 \| \leqslant \alpha$$

implies

$$\| x(t, t_0, x_0)\| < N, \qquad t \geqslant t_0 + T;$$

(B_4) *quasi-uniform-ultimately bounded* if the T in (B_3) is independent of t_0;

(B_5) *equi-ultimately bounded* if (B_1) and (B_3) hold at the same time;

(B_6) *uniform-ultimately bounded* if (B_2) and (B_4) hold simultaneously;

(B_7) *equi-Lagrange stable* if (B_1) and (S_7) hold simultaneously;

(B_8) *uniform-Lagrange stable* if (B_2) and (S_8) hold simultaneously;

PROPOSITION 3.13.1. If $f(t, 0) = 0$, $t \in J$, and β occurring in (B_1) and (B_2) has the property that $\beta \to 0$ as $\alpha \to 0$, then the definitions (B_1), (B_2) imply the definitions (S_1), (S_2), respectively.

The proof of the statement is obvious.

PROPOSITION 3.13.2. Quasi-equi-ultimate boundedness implies equi-boundedness if

$$\|f(t, x)\| \leqslant g(t, \| x \|), \tag{3.13.2}$$

where $g \in C[J \times R_+, R_+]$.

Proof. Consider the function $m(t) = \| x(t, t_0, x_0)\|$, where $x(t, t_0, x_0)$ is any solution of (3.13.1). Then,

$$\begin{aligned} D^+m(t) \leqslant \| x'(t, t_0, x_0)\| &= \|f(t, x(t, t_0, x_0))\| \\ &\leqslant g(t, m(t)), \end{aligned}$$

using assumption (3.13.2). By Theorem 1.4.1, we have

$$\| x(t, t_0, x_0)\| \leqslant r(t, t_0, \alpha), \qquad t \geqslant t_0, \tag{3.13.3}$$

whenever $\| x_0 \| \leqslant \alpha$, where $r(t, t_0, \alpha)$ is the maximal solution of

$$u' = g(t, u), \qquad u(t_0) = \alpha. \tag{3.13.4}$$

By the quasi-equi-ultimate boundedness, given $\alpha \geqslant 0$ and $t_0 \in J$, there exist two positive numbers N and $T = T(t_0, \alpha)$ such that the inequality $\| x_0 \| \leqslant \alpha$ implies

$$\| x(t, t_0, x_0)\| < N, \qquad t \geqslant t_0 + T.$$

Since $g(t, u) \geqslant 0$, the solution $r(t, t_0, \alpha)$ of (3.13.4) is monotonic nondecreasing in t, and therefore we have, from (3.13.3), that

$$\| x(t, t_0, x_0)\| \leqslant r(t_0 + T, t_0, \alpha), \qquad t \in [t_0, t_0 + T].$$

It then follows that

$$\| x(t, t_0, x_0) \| \leqslant \max[N, r(t_0 + T, t_0, \alpha)], \qquad t \geqslant t_0,$$

and this proves (B_1).

Analogous to the group of definitions (B_1)–(B_8), we can define the concepts of boundedness and Lagrange stability with respect to the scalar differential equation (3.2.3) and designate them by (B_1^*)–(B_8^*).

THEOREM 3.13.1. Assume that there exist functions $V(t, x)$ and $g(t, u)$ with the following properties:

(i) $g \in C[J \times R_+, R]$.

(ii) $V \in C[J \times R^n, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ is locally Lipschitzian in x, and, for $(t, x) \in J \times R^n$,

$$V(t, x) \geqslant b(\| x \|), \tag{3.13.5}$$

where $b \in \mathscr{K}$ on the interval $0 \leqslant u < \infty$ and $b(u) \to \infty$ as $u \to \infty$.

(iii) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times R^n$.

Then, the equiboundedness of Eq. (3.2.3) implies the equiboundedness of the system (3.13.1).

Proof. Let $\alpha \geqslant 0$ and $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \alpha$. In view of the hypotheses on $V(t, x)$, there exists a number $\alpha_1 = \alpha_1(t_0, \alpha)$ satisfying the inequalities

$$\| x_0 \| \leqslant \alpha, \qquad V(t_0, x_0) \leqslant \alpha_1$$

together. Assume that Eq. (3.2.3) is equibounded. Then, given $\alpha_1 \geqslant 0$ and $t_0 \in J$, there exists a $\beta_1 = \beta_1(t_0, \alpha)$ that is continuous in t_0 for each α such that

$$r(t, t_0, u_0) < \beta_1, \qquad t \geqslant t_0, \tag{3.13.6}$$

provided $u_0 \leqslant \alpha_1$. Moreover, as $b(u) \to \infty$ as $u \to \infty$, we can choose a $L = L(t_0, \alpha)$ verifying the relation

$$b(L) \geqslant \beta_1(t_0, \alpha). \tag{3.13.7}$$

Now let $u_0 = V(t_0, x_0)$. Then, assumption (iii) and Theorem 3.1.1 show that

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0, \tag{3.13.8}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.2.3). Suppose, if possible, that there is a solution $x(t, t_0, x_0)$ with $\| x_0 \| \leqslant \alpha$ having the property that, for some $t_1 > t_0$,

$$\| x(t_1, t_0, x_0)\| = L.$$

Then, because of relations (3.13.5), (3.13.6), (3.13.7), and (3.13.8), there results the following absurdity:

$$\begin{aligned} b(L) \leqslant V(t_1, x(t_1, t_0, x_0)) &\leqslant r(t_1, t_0, u_0) \\ &< \beta_1(t_0, \alpha) \leqslant b(L). \end{aligned}$$

The proof is complete, since this contradiction implies that (B_1) holds.

THEOREM 3.13.2. In addition to the hypotheses of Theorem 3.13.1, let $V(t, x)$ verify the inequality

$$V(t, x) \leqslant a(\| x \|), \tag{3.13.9}$$

where $a \in \mathscr{K}$ on the interval $0 \leqslant u < \infty$. Then, if Eq. (3.2.3) is uniform bounded, the system (3.13.1) is likewise uniform bounded.

Proof. The proof runs almost parallel to the proof of Theorem 3.13.1. We choose $\alpha_1 = a(\alpha)$, which is independent of t_0. Since $\beta_1 = \beta_1(\alpha)$ in this case, it is easy to see from the choice of L that it is also independent of t_0. Thus (B_2) is verified.

THEOREM 3.13.3. Under the assumptions of Theorem 3.13.1, the quasi-equi-ultimate boundedness of Eq. (3.2.3) implies the quasi-equi-ultimate boundedness of the system (3.13.1).

Proof. If $\alpha \geqslant 0$ and $t_0 \in J$ are given, then, as in the proof of Theorem 3.13.1, we can choose an $\alpha_1 = \alpha_1(t_0, \alpha)$ satisfying

$$\| x_0 \| \leqslant \alpha, \qquad V(t_0, x_0) \leqslant \alpha_1$$

at the same time. From the quasi-equi-ultimate boundedness of (3.2.3), given $\alpha_1 \geqslant 0$ and $t_0 \in J$, there exist positive numbers N_1 and $T = T(t_0, \alpha)$ such that

$$r(t, t_0, u_0) < N_1, \qquad t \geqslant t_0 + T \tag{3.13.10}$$

whenever $u_0 \leqslant \alpha_1$. Since $b(u) \to \infty$ with u, it is possible to find a positive number N verifying

$$b(N) \geqslant N_1. \tag{3.13.11}$$

We choose $u_0 = V(t_0, x_0)$ and obtain the estimate (3.13.8) as in Theorem 3.13.1. Now, let there exist a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$, $t_k \to \infty$ as $k \to \infty$ such that, for some solution $x(t, t_0, x_0)$ of (3.13.1) satisfying $\| x_0 \| \leqslant \alpha$, we have

$$\| x(t_k, t_0, x_0)\| \geqslant N.$$

We are led to the following contradiction, in view of relations (3.13.5), (3.13.8), (3.13.10), and (3.13.11):

$$\begin{aligned} b(N) \leqslant V(t_k, x(t_k, t_0, x_0)) &\leqslant r(t_k, t_0, u_0) \\ &< N_1 \leqslant b(N). \end{aligned}$$

This proves that the system (3.13.1) is quasi-equi-ultimately bounded.

THEOREM 3.13.4. Under the assumptions of Theorem 3.13.1, the equi-ultimate boundedness of Eq. (3.2.3) implies the equi-ultimate boundedness of the system (3.13.1).

The proof of this theorem can be constructed by combining the proofs of Theorems 3.13.1 and 3.13.3.

THEOREM 3.13.5. Let the hypotheses of Theorem 3.13.2 hold. Then, the quasi-uniform-ultimate boundedness of Eq. (3.2.3) assures the quasi-uniform-ultimate boundedness of the system (3.13.1).

Proof. As in Theorem 3.13.2, one can choose $\alpha_1 = a(\alpha)$ independent of t_0, and, consequently, quasi-uniform boundedness of Eq. (3.2.3) shows that $T = T(\alpha)$ is also independent of t_0. With these observations, the proof follows closely that of Theorem 3.13.3.

THEOREM 3.13.6. Let the hypotheses of Theorem 3.13.2 hold. Then the uniform-ultimate boundedness of Eq. (3.2.3) assures the uniform ultimate boundedness of the system (3.13.1).

The following two theorems present weakening of the conditions of Theorems 3.13.2 and 3.13.6. Let Z_ρ denote the set

$$Z_\rho = [x \in R^n : \| x \| \geqslant \rho].$$

THEOREM 3.13.7. Assume that there exist functions $V(t, x)$ and $g(t, u)$ fulfilling the following conditions:

(i) $g \in C[J \times R_+, R]$.

(ii) $V \in C[J \times Z_\rho, R_+]$, $V(t, x)$ is locally Lipschitzian in x and satisfies

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times Z_\rho, \tag{3.13.12}$$

where $a(u)$, $b(u) > 0$ are continuous and increasing for $u \geqslant \rho$, and

$$b(u) \to \infty \qquad \text{as} \quad u \to \infty;$$

(iii) $D^+V(t, x) \leqslant g(t, V(t, x))$, $(t, x) \in J \times Z_\rho$.

Then, the uniform boundedness of Eq. (3.2.3) implies the uniform boundedness of the system (3.13.1).

Proof. Let $\alpha > 0$ (we may suppose $\alpha > \rho$) and $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \alpha$. Define $\alpha_1 = a(\alpha)$. From the uniform boundedness of (3.2.3), it follows that

$$r(t, t_0, u_0) < \beta_1(\alpha), \qquad t \geqslant t_0, \tag{3.13.13}$$

provided $u_0 \leqslant \alpha_1$. Since $b(u) \to \infty$ as $u \to \infty$, there exists a $\beta = \beta(\alpha)$ such that

$$b(\beta) \geqslant \beta_1(\alpha). \tag{3.13.14}$$

Now, if we suppose that, for some solution $x(t, t_0, x_0)$ of (3.13.1) with $\| x_0 \| \leqslant \alpha$, we have

$$\| x(t_1, t_0, x_0)\| = \beta \qquad \text{at} \quad t = t_1 > t_0,$$

then there exists a $t_2 < t_1$ satisfying

$$\| x(t_2, t_0, x_0)\| = \alpha$$

and

$$\rho < \alpha \leqslant \| x(t, t_0, x_0)\| \leqslant \beta, \qquad t \in [t_2, t_1]. \tag{3.13.15}$$

Considering the function $V(t, x(t, t_0, x_0))$, it follows that

$$V(t_1, x(t_1, t_0, x_0)) \geqslant b(\beta).$$

Choose $u_0 = a(\| x_2 \|)$, where $x_2 = x(t_2, t_0, x_0)$. Then, condition (iii) and Theorem 3.1.1 show that, because of (3.13.15),

$$V(t, y(t, t_2, x_2)) \leqslant r(t, t_2, u_0), \qquad t \in [t_2, t_1], \tag{3.13.16}$$

where $y(t, t_2, x_2)$ is any solution through (t_2, x_2) of (3.13.1). Thus, (3.13.16) is true for $x(t, t_0, x_0)$ on the interval $t_2 \leqslant t \leqslant t_1$. We therefore obtain, from the foregoing considerations, using (3.13.13), (3.13.14), and (3.13.16),

$$b(\beta) \leqslant V(t_1, x(t_1, t_0, x_0)) \leqslant r_1(t_1, t_2, u_0) < \beta_1(\alpha) \leqslant b(\beta).$$

This contradiction proves (B_2), and the proof of the theorem is complete.

COROLLARY 3.13.1. The function $g(t, u) \equiv 0$ is admissible in Theorem 3.13.7.

Proof. Let $\alpha > 0$ and $t_0 \in J$ be given as before, and let $\| x_0 \| \leqslant \alpha$. We choose $\beta = \beta(\alpha)$ to satisfy the relation

$$b(\beta) > a(\alpha). \tag{3.13.17}$$

The assumption that (B_2) does not hold for some solution $x(t, t_0, x_0)$ with $\| x_0 \| \leqslant \alpha$ implies, as before, the inequality (3.13.15), and consequently, from (3.13.13), we have

$$V(t_2, x(t_2, t_0, x_0)) \leqslant a(\alpha)$$

and

$$b(\beta) \leqslant V(t_1, x(t_1, t_0, x_0)).$$

On the other hand, by condition (iii), it follows that

$$V(t_2, x(t_2, t_0, x_0)) \leqslant V(t_1, x(t_1, t_0, x_0)),$$

since the function $V(t, x(t, t_0, x_0))$ is nonincreasing in t. The foregoing inequalities lead to a contradiction, in view of (3.13.17), thus proving that (B_2) holds.

THEOREM 3.13.8. Under the hypotheses of Theorem 3.13.7, if Eq. (3.2.3) is uniformly ultimately bounded, then the system (3.13.1) is likewise uniformly ultimately bounded.

Proof. By Theorem 3.13.7, the system (3.13.1) is uniformly bounded. Hence, there is a positive number B such that, if $\| x_0 \| \leqslant \rho$, $\| x(t, t_0, x_0) \| < B$, $t \geqslant t_0$.

Let now $\alpha > \rho$ and $t_0 \in J$ be given, and let $\rho \leqslant \| x_0 \| \leqslant \alpha$. Define $\alpha_1 = a(\alpha)$. From definition (B_4^*), it follows that, given $\alpha_1 \geqslant 0$, $t_0 \in J$, there exist positive numbers N_1 and $T = T(\alpha)$ such that

$$r(t, t_0, u_0) < N_1, \qquad t \geqslant t_0 + T, \tag{3.13.18}$$

provided $u_0 \leqslant \alpha_1$. Let $N^* = \max(N, B)$, where N is chosen so as to satisfy the inequality

$$b(N) \geqslant N_1. \tag{3.13.19}$$

Clearly, $N^* > \rho$, and the choice of N is possible since $b(u) \to \infty$ as $u \to \infty$. We claim that, with this N^* and $T(\alpha)$, definition (B_4) holds. Suppose that this is false. Since the solutions $x(t, t_0, x_0)$ starting in

$\| x_0 \| \leqslant \rho$ remain in $\| x \| < N$, it is enough to consider only those solutions $x(t, t_0, x_0)$ which start in $\rho \leqslant \| x_0 \| \leqslant \alpha$. If $u_0 = a(\| x_0 \|)$, assumption (iii) yields, for such solutions, the inequality

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0. \tag{3.13.20}$$

Let there exist a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$, $t_k \to \infty$ as $k \to \infty$, such that $\| x(t_k, t_0, x_0) \| \geqslant N^*$ for some solution starting in $\rho \leqslant \| x_0 \| \leqslant \alpha$. Then, the following inequality results, using relations (3.13.18), (3.13.19), and (3.13.20):

$$b(N^*) \leqslant V(t_k, x(t_k, t_0, x_0)) \leqslant r(t_k, t_0, u_0) < N_1 \leqslant b(N),$$

whence we have $N^* < N$. This is absurd in view of the definition of N^*, since $N^* \geqslant N$. The proof is therefore complete.

COROLLARY 3.13.2. The function $g(t, u) = -C(u)$, $C \in \mathscr{K}$, is admissible in Theorem 3.13.8.

With this choice of $g(t, u)$, evidently (B_6^*) holds, and hence the corollary is a consequence of Theorem 3.13.8.

COROLLARY 3.13.3. The replacement of assumption (iii) in Theorem 3.13.7 by

$$D^+V(t, x) \leqslant -\alpha(\| x \|), \qquad (t, x) \in J \times Z_\rho,$$

where $\alpha \in \mathscr{K}$, is also admissible.

Using the right inequality of (3.13.12), it follows that

$$D^+V(t, x) \leqslant -\alpha[a^{-1}V(t, x)] = -C(V(t, x)), \qquad C \in \mathscr{K},$$

and hence the truth of this corollary follows from Corollary 3.13.2.

THEOREM 3.13.9. Let the assumptions of Theorem 3.13.1 hold. Then, equi-Lagrange stability of Eq. (3.2.3) assures the equi-Lagrange stability of the system (3.13.1).

Proof. By Theorem 3.13.1, equiboundedness of the system (3.13.1) follows, and hence (S_7) remains to be proved. Let $\epsilon > 0$, $\alpha \geqslant 0$, and $t_0 \in J$ be given, and let $\| x_0 \| \leqslant \alpha$. As in the proof of Theorem 3.13.1, there exists an $\alpha_1 = \alpha_1(t_0, \alpha)$ satisfying

$$\| x_0 \| \leqslant \alpha, \qquad V(t_0, x_0) \leqslant \alpha_1$$

simultaneously. Since (S_7^*) holds, given $\alpha_1 \geqslant 0$, $b(\epsilon) > 0$, and $t_0 \in J$, there exists a $T = T(t_0, \epsilon, \alpha)$ such that $u_0 \leqslant \alpha_1$ implies

$$r(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 + T. \tag{3.13.21}$$

Choose $u_0 = V(t_0, x_0)$. Then, condition (iii) and Theorem 3.1.1 yield the inequality (3.13.8). If possible, let there exist a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$, $t_k \to \infty$ as $k \to \infty$, such that, for some solution $x(t, t_0, x_0)$ satisfying $\| x_0 \| \leqslant \alpha$, we have

$$\| x(t_k, t_0, x_0)\| \geqslant \epsilon.$$

This implies, in view of the inequalities (3.13.5), (3.13.8), and (3.13.21), the following absurdity:

$$b(\epsilon) \leqslant V(t_k, x(t_k, t_0, x_0)) \leqslant r(t_k, t_0, u_0) < b(\epsilon)$$

which proves (S_7). The proof is complete.

THEOREM 3.13.10. Let the assumptions of Theorem 3.13.2 hold. Then, the uniform Lagrange stability of Eq. (3.2.3) implies the uniform Lagrange stability of the system (3.13.1).

Proof. Since uniform boundedness of the system (3.13.1) follows from Theorem 3.13.2, definition (S_8) needs to be proved. Choosing $\alpha_1 = a(\alpha)$ and following the proof of Theorem 3.13.9, we observe that $T = T(\epsilon, \alpha)$ is independent of t_0 because of (S_8^*). This shows that (S_8) holds, and the theorem is proved.

Using the similar techniques as employed in Sect. 3.6, we can construct Lyapunov functions in the case of boundedness also. The following theorem is a typical result in that direction.

THEOREM 3.13.11. Assume that:

(i) the function $f \in C[J \times R^n, R^n]$, $\partial f(t, x)/\partial x$ exists and is continuous for $(t, x) \in J \times R^n$, and

$$\beta_1(\| x_0 \|) \leqslant \| x(t, 0, x_0)\| \leqslant \beta_2(\| x_0 \|), \qquad t \geqslant 0, \tag{3.13.22}$$

where $\beta_1(u), \beta_2(u) > 0$ are continuous and increasing for $u \geqslant 0$, and $\beta_1(u) \to \infty$ as $u \to \infty$;

(ii) the function $g \in C[J \times R_+, R]$, $\partial g(t, u)/\partial u$ exists and is continuous for $(t, u) \in J \times R_+$, and

$$\gamma_1(u_0) \leqslant u(t, 0, u_0) \leqslant \gamma_2(u_0), \qquad t \geqslant 0, \tag{3.13.23}$$

where $\gamma_1(u), \gamma_2(u) > 0$ are continuous and increasing for $u \geqslant 0$ and $\gamma_1(u) \to \infty$ as $u \to \infty$.

Then, there exists a function $V(t, x)$ and a constant $\rho > 0$ satisfying the following conditions:

(1) $V \in C[J \times Z_\rho, R_+]$, $V(t, x)$ possesses continuous partial derivatives with respect to t and the components of x, and

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times Z_\rho,$$

where $a(u), b(u) > 0$ are continuous and increasing for $u \geqslant 0$ and $b(u) \to \infty$ as $u \to \infty$.

(2) $$V'(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) = g(t, V(t, x)), (t, x) \in J \times Z_\rho.$$

Proof. Let $x(t, 0, x_0)$, $u(t, 0, u_0)$ be the solutions of (3.13.1), (3.2.3) through $(0, x_0)$, $(0, u_0)$ satisfying (3.13.22), (3.13.23), respectively. Under the assumptions of the theorem, the continuity and differentiability of the solutions $x(t, t_0, x_0)$, $u(t, t_0, u_0)$ with respect to their arguments is guaranteed. Define $\rho = \beta_2(0)$, and observe that the common domain of definition of the inverse functions $\beta_1^{-1}, \beta_2^{-1}$ is $[\rho, \infty)$. Hence, denoting $x(t, 0, x_0)$ by x, so that $x_0 = x(0, t, x)$, we get from (3.13.22) the inequality

$$\beta_2^{-1}(\| x \|) \leqslant \| x(0, t, x) \| \leqslant \beta_1^{-1}(\| x \|), \qquad x \in Z_\rho.$$

We choose a continuous function $\mu(x)$ for $x \in R^n$, possessing continuous partial derivatives $\partial\mu(x)/\partial x$ there and such that

$$\alpha_1(\| x \|) \leqslant \mu(x) \leqslant \alpha_2(\| x \|),$$

where $\alpha_1, \alpha_2 \in \mathscr{K}$ on the interval $0 \leqslant u < \infty$, and $\alpha_1(u) \to \infty$ as $u \to \infty$. For $x \in Z_\rho$, define the function

$$V(t, x) = u(t, 0, \mu(x(0, t, x))).$$

Then, it is easy to obtain

$$V'(t, x) = g(t, V(t, x)), (t, x) \in J \times Z_\rho.$$

Furthermore, if $\| x \| \geqslant \rho$, we have

$$\begin{aligned} V(t, x) &\geqslant \gamma_1[\mu(x(0, t, x))] \\ &\geqslant \gamma_1[\alpha_1(\| x(0, t, x) \|)] \\ &\geqslant \gamma_1[\alpha_1(\beta_2^{-1}(\| x \|))] \equiv b(\| x \|) \end{aligned}$$

and

$$\begin{aligned} V(t, x) &\leqslant \gamma_2[\mu(x(0, t, x))] \\ &\leqslant \gamma_2[\alpha_2(\| x(0, t, x)\|)] \\ &\leqslant \gamma_2[\alpha_2(\beta_1^{-1}(\| x \|))] = a(\| x \|). \end{aligned}$$

Clearly $a(u), b(u) > 0$ are continuous and increasing for $u \geqslant \rho$, and $b(u) \to \infty$ as $u \to \infty$. The theorem is proved.

3.14. Eventual stability

We shall now consider a notion that is a generalization of Lyapunov's stability. Let $x(t, t_0, x_0)$ be any solution of (3.2.1).

DEFINITION 3.14.1. The set $x = 0$ is said to be [with respect to the system (3.2.1)]:

(E_1) *eventually uniformly stable* if, for every $\epsilon > 0$, there exist a $\delta = \delta(\epsilon) > 0$ and $\tau = \tau(\epsilon) > 0$ such that

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0 \geqslant \tau(\epsilon),$$

provided $\| x_0 \| \leqslant \delta$;

(E_2) *eventually quasi-uniformly asymptotically stable* if, for every $\epsilon > 0$, there exist positive numbers δ_0, τ_0, and $T = T(\epsilon)$ such that

$$\| x(t, t_0, x_0)\| < \epsilon, \qquad t \geqslant t_0 + T, \qquad t_0 \geqslant \tau_0,$$

provided $\| x_0 \| \leqslant \delta_0$;

(E_3) *eventually uniformly asymptotically stable* if (E_1) and (E_2) hold simultaneously;

(E_4) *eventually exponentially asymptotically stable* if there exist constants $L > 0$, $\alpha > 0$ such that

$$\| x(t, t_0, x_0)\| \leqslant L\| x_0 \| \exp[-\alpha(t - t_0)], \qquad t \geqslant t_0, \tag{3.14.1}$$

provided $0 < r < \| x_0 \| < \rho$ and $t_0 \geqslant \theta(r)$, where $\theta(r)$ is a monotonic decreasing function of r for $0 < r < \rho$.

REMARK 3.14.1. Notice that, if (E_1) holds and if $x = 0$ is a trivial solution of (3.2.1), then the uniform Lyapunov stability (S_2) results from the continuity of solutions with respect to the initial values, provided the unicity of solutions of (3.2.1) is assured. Similarly, (E_3)

implies, in such a case, uniform asymptotic stability of the trivial solution of (3.2.1).

As usual, let us denote by (E_1^*)–(E_4^*) the corresponding notions of the set $u = 0$ with respect to the differential equation (3.2.3).

THEOREM 3.14.1. Assume that there exist functions $V(t, x)$ and $g(t, u)$ verifying the following properties:

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is locally Lipschitzian in x, and

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|),$$

for $0 < r < \| x \| < \rho$ and $t \geqslant \theta(r)$, where $a, b \in \mathscr{K}$ and $\theta(r)$ is continuous and monotonic decreasing in r for $0 < r < \rho$.

(ii) $g \in C[J \times R_+, R]$, and the set $u = 0$ is eventually uniformly stable with respect to (3.2.3).

(iii) $f \in C[J \times S_\rho, R^n]$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)),$$

for $0 < r < \| x \| < \rho$ and $t \geqslant \theta(r)$.

Then, the set $x = 0$ is eventually uniformly stable with respect to the system (3.2.1).

Proof. Let $0 < \epsilon < \rho$. Since the set $u = 0$ is eventually uniformly stable, given $b(\epsilon) > 0$, there exist a $\delta_1 = \delta_1(\epsilon) > 0$ and $\tau_1 = \tau_1(\epsilon) > 0$ such that

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 \geqslant \tau_1(\epsilon), \tag{3.14.2}$$

if $u_0 \leqslant \delta_1$. We define $\delta = a^{-1}(\delta_1)$ and $\tau_2(\epsilon) = \theta(\delta(\epsilon))$. Let $\tau = \tau(\epsilon) = \max[\tau_1(\epsilon), \tau_2(\epsilon)]$. Then, (E_1) holds with this choice of $\delta(\epsilon)$ and $\tau(\epsilon)$. If this were not true, there exist numbers t_1, t_2 such that $t_2 > t_1 > t_0 \geqslant \tau$,

$$\| x(t_1, t_0, x_0)\| = \delta, \qquad \| x(t_2, t_0, x_0)\| = \epsilon,$$

and

$$\delta < \| x(t, t_0, x_0)\| < \epsilon, \qquad t \in (t_1, t_2).$$

Choose $u_0 = a(\| x_1 \|)$, where $x_1 = x(t_1, t_0, x_0)$. Then, condition (iii) and Theorem 3.11.1 show that

$$V(t, y(t, t_1, x_1)) \leqslant r(t, t_1, u_0), \qquad t \in [t_1, t_2], \tag{3.14.3}$$

where $y(t, t_1, x_1)$ is any solution of (3.2.1) through (t_1, x_1), $r(t, t_1, u_0)$ being the maximal solution of (3.2.3) through (t_1, u_0). Thus, (3.14.3)

is also true for $x(t, t_0, x_0)$ on the interval $t_1 \leqslant t \leqslant t_2$. We therefore obtain

$$b(\epsilon) \leqslant V(t_2, x(t_2, t_0, x_0)) \leqslant r(t_2, t_1, u_0) < b(\epsilon),$$

taking into account the uniformity of the relation (3.14.2) and the fact $t_2 > t_1 > t_0 \geqslant \tau$. This absurdity that we are led to prove (E_1) is true, and the proof is complete.

COROLLARY 3.14.1. If, instead of the eventual uniform stability of the set $u = 0$, it is assumed that the trivial solution $u = 0$ is uniformly stable, the conclusion of Theorem 3.14.1 remains the same. In particular, $g(t, u) \equiv 0$ is admissible.

THEOREM 3.14.2. Suppose that there exists a function $V(t, x)$ such that

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is Lipschitzian in x for a constant $L > 0$, and

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|),$$

for $0 < r < \| x \| < \rho$ and $t \geqslant \theta(r)$, where $a, b \in \mathscr{K}$ and $\theta(r)$ is continuous and monotonic decreasing in r for $0 < r < \rho$;

(ii) $f \in C[J \times S_\rho, R^n]$, and $D^+V(t, x) \leqslant 0$ for $0 < r < \| x \| < \rho$ and $t \geqslant \theta(r)$.

Then, the set $x = 0$ is eventually uniformly stable with respect to the perturbed system

$$x' = f(t, x) + R(t, x), \tag{3.14.4}$$

where $R \in C[J \times S_\rho, R^n]$, and, for every continuous function $x(t)$ such that $\| x(t) \| \leqslant \rho^* < \rho$, $t \geqslant 0$,

$$\int_0^\infty \| R(s, x(s)) \| \, ds < \infty.$$

Proof. For a given $\epsilon > 0$, $\epsilon \leqslant \rho^*$, we choose a $\delta(\epsilon) > 0$ so that

$$2a(\delta) < b(\epsilon), \qquad \tau_1(\epsilon) = \theta(\delta(\epsilon)). \tag{3.14.5}$$

Let $\phi(t) = \max_{\| x \| \leqslant \rho^*} \| R(t, x) \|$. Then, since $\phi(t)$ is integrable, there exists a $\tau_2(\epsilon) > 0$ such that, if $t_0 \geqslant \tau_2(\epsilon)$, we have

$$\int_{t_0}^\infty \phi(s) \, ds < a(\delta)/L,$$

where L is the Lipschitz constant for $V(t, x)$. Let $\tau(\epsilon) = \max[\tau_1(\epsilon), \tau_2(\epsilon)]$. Then (E_1) is true with $\tau(\epsilon)$ and $\delta(\epsilon)$. For otherwise, there would exist a $t_1 > t_0 \geqslant \tau(\epsilon)$ such that

$$\| x(t_1)\| = \epsilon, \qquad \| x(t)\| \leqslant \epsilon \leqslant \rho^*, \qquad t \in [t_0, t_1],$$

where $x(t) = x(t, t_0, x_0)$ is some solution of (3.2.1). This implies, setting $m(t) = V(t, x(t))$, the inequality

$$D^+m(t) \leqslant L\phi(t),$$

and, consequently,

$$m(t) \leqslant m(t_0) + L \int_{t_0}^{t} \phi(s)\, ds, \qquad t \geqslant t_0 .$$

Hence, at $t = t_1$, there results

$$b(\epsilon) \leqslant m(t_1) = V(t_1, x(t_1)) \leqslant V(t_0, x_0) + L \int_{t_0}^{t_1} \phi(s)\, ds$$
$$\leqslant a(\delta) + a(\delta) = 2a(\delta),$$

which is a contradiction in view of (3.14.5). The proof is complete.

THEOREM 3.14.3. Suppose that there exists a function $V(t, x)$ such that

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is locally Lipschitzian in x, and

$$b(\| x \|) \leqslant V(t, x) \leqslant a(\| x \|)$$

for $0 < r \leqslant \| x \| < \rho$ and $t \geqslant \theta(r)$, where $a, b \in \mathscr{K}$ and $\theta(r)$ is continuous and monotonic decreasing in r for $0 < r < \rho$;

(ii) $f \in C[J \times S_\rho, R^n]$, and

$$D^+V(t, x) \leqslant -C(\| x \|)$$

for $0 < r \leqslant \| x \| < \rho$, $t \geqslant \theta(r)$, and $C \in \mathscr{K}$.

Then, the set $x = 0$ is eventually uniformly asymptotically stable.

Proof. By Corollary 3.14.1, eventual uniform stability of the set $x = 0$ follows.

Let $\epsilon > 0$ be given. Designate $\delta_0 = \delta(\rho)$, $\tau_0 = \tau(\rho)$, and $T(\epsilon) = \tau(\epsilon) + a(\rho)/C[\delta(\epsilon)]$. Let $t_0 \geqslant \tau_0$ and $\| x_0 \| \leqslant \delta_0$.

To prove the theorem, it is sufficient to show that there exists a $t^* \in [t_0 + \tau(\epsilon), t_0 + T(\epsilon)]$ such that $\| x(t^*, t_0, x_0)\| < \delta(\epsilon)$, because the set $x = 0$ is eventually uniformly stable. Assume, if possible, that

$$\delta(\epsilon) \leqslant \| x(t, t_0, x_0)\| < \rho, \qquad t \in [t_0 + \tau(\epsilon), t_0 + T(\epsilon)].$$

Using assumptions (i) and (ii), we get

$$D^+V(t, x(t, t_0, x_0)) \leqslant -C(\| x(t, t_0, x_0)\|) \leqslant -C[\delta(\epsilon)],$$

for $t \in [t_0 + \tau(\epsilon), t_0 + T(\epsilon)]$, which, by integration, yields

$$\begin{aligned} &V(t_0 + T(\epsilon), x(t_0 + T(\epsilon), t_0, x_0)) \\ &\qquad \leqslant V(t_0 + \tau(\epsilon), x(t_0 + \tau(\epsilon), t_0, x_0)) - C[\delta(\epsilon)][T(\epsilon) - \tau(\epsilon)]. \end{aligned}$$

It then follows, from this relation, that

$$\begin{aligned} 0 < b(\delta(\epsilon)) &\leqslant V(t_0 + T(\epsilon), x(t_0 + T(\epsilon), t_0, x_0)) \\ &\leqslant a(\| x(t_0 + \tau(\epsilon), t_0, x_0)\|) - C[\delta(\epsilon)] \frac{a(\rho)}{C[\delta(\epsilon)]} \\ &\leqslant a(\rho) - a(\rho) = 0. \end{aligned}$$

This contradiction implies that there exists a $t^* \in [t_0 + \tau(\epsilon), t_0 + T(\epsilon)]$ such that $\| x(t^*, t_0, x_0)\| < \delta(\epsilon)$ and proves the theorem.

The following converse theorem for eventual uniform asymptotic stability may be proved, in the same way as in Theorem 3.6.9.

THEOREM 3.14.4. Assume that the set $x = 0$ is eventually uniformly asymptotically stable with respect to the system (3.2.1). Suppose that

$$\|f(t, x) - f(t, y)\| \leqslant L(t)\| x - y \|,$$

for $t \geqslant 0$, $x, y \in S_\rho$, and

$$\int_t^{t+u} L(s)\, ds \leqslant Lu, \qquad u \geqslant 0.$$

Then, there exist functions $a, b, C \in \mathscr{K}$, $\theta(u)$, and $V(t, x)$ satisfying

(1) $V \in C[J \times S_\rho, R_+]$ and

$$| V(t, x) - V(t, y)| \leqslant M(r)\| x - y \|$$

for $r > 0, r \leqslant \| x \| \leqslant \delta_0$, $r \leqslant \| y \| \leqslant \delta_0$, $t \geqslant \tau_0$, $M(r)$ being continuous and decreasing on $(0, \delta_0)$;

(2) $b(\| x \|) \leqslant V(t, x)$, $t \geqslant \tau_0$, $\| x \| \leqslant \delta_0$;

(3) $V(t, x) \leqslant a(\| x \|)$, $0 < r \leqslant \| x \| \leqslant \delta_0$, $t \geqslant \theta(r)$, where $\theta(u)$ is continuous and decreasing in u for $0 < u < \rho$;

(4) $D^+V(t, x) \leqslant -C(\| x \|)$, $t \geqslant \tau_0$, $\| x \| \leqslant \delta_0$.

Proof. As in the case of uniform asymptotic stability, we can find $\delta(\epsilon)$, $\tau(\epsilon)$, and $T(\epsilon)$ such that $\delta \in \mathscr{K}$ and $\tau, T \in \mathscr{L}$. Let $G(r)$ be the same function as in Theorem 3.6.9. Then, we define

$$V(t, x) = \sup_{\sigma \geqslant 0} G(\| x(t + \sigma, t, x)\|) \frac{1 + \alpha\sigma}{1 + \sigma},$$

for $t \geqslant \tau_0$, $\| x \| < \delta_0$. Clearly, $V(t, x) \geqslant G(\| x \|)$, and therefore $b(u) = G(u)$. Let $\epsilon(\delta)$ be the inverse function of $\delta(\epsilon)$, and let $\theta(u) = \tau(\epsilon(u))$. We have

$$\| x(t + \sigma, t, x)\| < \epsilon(\| x \|), \qquad 0 < r \leqslant \| x \| \leqslant \delta_0, \qquad t \geqslant \theta(r).$$

As a consequence,

$$\frac{1 + \alpha\sigma}{1 + \sigma} G(\| x(t + \sigma, t, x)\|) \leqslant \alpha G(\epsilon(\| x \|)),$$

for $0 < r \leqslant \| x \| \leqslant \delta_0$, $t \geqslant \theta(r)$, and thus it results that

$$V(t, x) \leqslant a(\| x \|), \qquad 0 < r \leqslant \| x \| \leqslant \delta_0, \qquad t \geqslant \theta(r),$$

where $a(u) = \alpha G(\epsilon(u))$. The rest of the proof is similar to that of Theorem 3.6.9. We omit the details.

THEOREM 3.14.5. Let the assumptions of Theorem 3.14.4 hold. Assume that the perturbation $R(t, x)$ obeys

$$\| R(t, x)\| \leqslant \sigma(t), \qquad \sigma \in \mathscr{L}, \qquad \| x \| \leqslant \delta_0. \tag{3.14.6}$$

Then, the set $x = 0$ is eventually uniformly asymptotically stable with respect to the system (3.14.4).

Proof. Let $V(t, x)$ be the function constructed as in Theorem 3.14.4. For $0 < r < \| x \| < \delta_0$, $t \geqslant \theta(r)$, and $h > 0$ sufficiently small, we have

$$\begin{aligned} &V(t + h, x + h[f(t, x) + R(t, x)]) - V(t, x) \\ &\qquad \leqslant M(r)h\| R(t, x)\| + V(t + h, x + hf(t, x)) - V(t, x), \end{aligned}$$

and hence, by (3.14.6),

$$\begin{aligned} D^+V(t, x)_{(3.14.4)} &\leqslant D^+V(t, x)_{(3.2.1)} + M(r)\| R(t, x)\| \\ &\leqslant -C(\| x \|) + M(r)\sigma(t). \end{aligned}$$

Since $\sigma \in \mathscr{L}$, there exists a $\theta_1 \in \mathscr{L}$ such that

$$\sigma(t) \leqslant \frac{C(r)}{2M(r)} \qquad \text{if} \quad t \geqslant \theta_1(r).$$

It then follows that

$$D^+V(t, x)_{(3.14.4)} \leqslant -\tfrac{1}{2}C(\| x \|),$$

provided $0 < r < \| x \| < \delta_0$ and $t \geqslant \theta_2(r)$, where $\theta_2(r) = \max[\theta(r), \theta_1(r)]$. The conclusion now follows from Theorem 3.14.3.

THEOREM 3.14.6. Assume that the set $x = 0$ is eventually exponentially asymptotically stable with respect to the system (3.2.1) and that $f(t, x)$ is linear in x. Then, there exists a function $V(t, x)$ satisfying

(1) $V \in C[J \times S_\rho, R_+]$, and $V(t, x)$ is Lipschitzian in x for a constant $L > 0$;

(2) $\| x \| \leqslant V(t, x) \leqslant L \| x \|$, $0 < r \leqslant \| x \| < \rho$, $t \geqslant \theta(r)$, where $\theta(r)$ is the same function defined in (E_4);

(3) $D^+V(t, x)_{(3.2.1)} \leqslant -\alpha V(t, x)$, $0 < r \leqslant \| x \| < \rho$, $t \geqslant \theta(r)$.

Proof. We define the function

$$V(t, x) = \sup_{\sigma \geqslant 0} \| x(t + \sigma, t, x) \| e^{\alpha\sigma},$$

for $0 < r \leqslant \| x \| < \rho$ and $t \geqslant \theta(r)$. Following the proof of Theorem 3.6.1, it is easy to prove this theorem with appropriate changes.

THEOREM 3.14.7. Suppose that the hypotheses of Theorem 3.14.6 hold. Let the perturbation $R(t, x)$ satisfy

$$\| R(t, x) \| \leqslant \beta \| x \|$$

for $0 < r \leqslant \| x \| < \rho$ and $t \geqslant \theta_1(r)$, where $\theta_1(u)$ is continuous, decreasing in u for $0 < u < \rho$, and β is sufficiently small. Then, the set $x = 0$ is eventually exponentially asymptotically stable with respect to the system (3.14.4).

Proof. By hypotheses, there exists a function $V(t, x)$ satisfying the properties (1), (2), and (3) of Theorem 3.14.6. For $0 < r < \| x \| < \rho$, $t \geqslant \theta_2(r)$, and $h > 0$ sufficiently small, we deduce that

$$\begin{aligned} D^+ V(t, x) &\leqslant -\alpha V(t, x) + L \| R(t, x) \| \\ &\leqslant -\alpha V(t, x) + L\beta \| x \| \\ &\leqslant (-\alpha + L\beta) V(t, x) \\ &\leqslant -\gamma V(t, x), \end{aligned}$$

where $\gamma > 0$, $\gamma < \alpha - L\beta$, and $\theta_2(r) = \max[\theta(r), \theta_1(r)]$. This choice of γ is possible, since β is sufficiently small. The stated result is evident from the preceding inequality, because of the properties of $V(t, x)$.

3.15. Asymptotic behavior

We shall discuss a number of results dealing with the asymptotic behavior of solutions. Let us begin with the following.

THEOREM 3.15.1. Let there exist functions $V(t, x)$ and $g(t, u)$ fulfilling the following conditions:

(i) $V \in C[J \times R^n, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ is Lipschitzian in x for a continuous function $K(t) \geqslant 0$, and

$$b(\|x\|) \leqslant V(t, x),$$

where $b \in \mathscr{K}$ such that $b(u) \to \infty$ as $u \to \infty$.

(ii) $g \in C[J \times R_+, R]$, and

$$\begin{aligned} D^+V(t, x - y) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x - y + h[(f(t, x) - f(t, y)]) \\ &\qquad - V(t, (x - y))] \\ &\leqslant g(t, V(t, x - y)), \end{aligned}$$

where $f \in C[J \times R^n, R^n]$.

(iii) Every solution $u(t)$ of

$$u' = g(t, u) + K(t)\|f(t, 0)\|, \qquad u(t_0) = u_0 \geqslant 0, \tag{3.15.1}$$

tends to zero as $t \to \infty$.

Then, every solution $x(t)$ of the system

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{3.15.2}$$

tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be any solution of (3.15.2) such that $V(t_0, x_0) \leqslant u_0$. Consider the function

$$m(t) = V(t, x(t)).$$

For small $h > 0$, we have

$$m(t+h) = V(t+h, x(t) + hf(t, x) + \epsilon(h))$$
$$\leqslant K(t)h\left[\|f(t, 0)\| + \frac{\|\epsilon(h)\|}{h}\right]$$
$$+ V(t+h, x(t) + h[f(t, x(t)) - f(t, 0)]),$$

where $\epsilon(h)/h \to 0$ as $h \to 0$. Employing assumption (ii) with $y = 0$, we arrive at the differential inequality

$$D^+m(t) \leqslant g(t, m(t)) + K(t)\|f(t, 0)\|,$$

and this yields, by Theorem 1.4.1, the estimate

$$V(t, x(t)) \leqslant r(t), \qquad t \geqslant t_0\,,$$

$r(t)$ being the maximal solution of (3.15.1). The statement of the theorem is now a direct consequence in view of conditions (i) and (iii).

THEOREM 3.15.2. Assume that there exist functions $V(t, x)$ and $g(t, u)$ enjoying the following properties:

(i) $V \in C[J \times R^n, R_+]$, $V(t, 0) \equiv 0$, and $V(t, x)$ is positive definite and locally Lipschitzian in x.

(ii) $g \in C[J \times R_+, R]$, and all solutions $u(t) = u(t, t_0, u_0)$, $0 \leqslant u_0 \leqslant \alpha$, of (3.2.3) have the property that $\lim_{t\to\infty} u(t) = 0$.

(iii) The function $D^+V(t, x)$ satisfies the inequality

$$D^+V(t, x) \leqslant g(t, V(t, x))$$

for $t \in J$ and $x \in Z$, where Z is the set defined by

$$Z = [x \in R^n; r(t) < V(t, x) < r(t) + \epsilon_0\,, t \geqslant t_0],$$

$r(t)$ being the maximal solution of (3.2.3) and ϵ_0 a certain small positive number.

Then, the domain of attraction for the solutions of (3.15.2) is the set

$$\Omega = [x \in R^n : V(t, x) \leqslant \alpha, t \in J],$$

that is, all solutions $x(t)$ of (3.15.2) such that $x_0 \in \Omega$ tend to zero as $t \to \infty$.

Proof. If $x(t)$ is any solution of (3.15.2) such that $x_0 \in \Omega$, we choose $u_0 = V(t_0, x_0)$ and obtain, by Corollary 3.1.2, the estimate

$$V(t, x(t)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0 .$$

The positive definiteness of $V(t, x)$ and assumption (ii) imply the stated result.

THEOREM 3.15.3. Assume that

(i) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $\partial f(t, x)/\partial x$ exists and is continuous on $J \times S_\rho$;

(ii) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ is Lipschitzian in x for a constant $K_2 > 0$, and

$$K_1\| x \| \leqslant V(t, x), \qquad K_1 > 0, \qquad (t, x) \in J \times S_\rho ;$$

(iii)

$$\begin{aligned} D^+V(t, x) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf_x(t, 0)x) - V(t, x)] \\ &\leqslant \alpha(t)V(t, x), \qquad (t, x) \in J \times S_\rho , \end{aligned}$$

where $\alpha \in C[J, R]$, and

$$\limsup_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \alpha(s)\, ds = \sigma_0 < 0. \tag{3.15.3}$$

Then, the trivial solution of (3.15.2) is asymptotically stable.

Proof. By assumption (i), given $\epsilon > 0$, it is possible to find a $\delta(\epsilon) > 0$ such that

$$f(t, x) = f_x(t, 0)x + F(t, x),$$

where

$$\| F(t, x)\| \leqslant \epsilon \| x \| \qquad \text{if} \quad \| x \| \leqslant \delta(\epsilon), \tag{3.15.4}$$

uniformly in t. It then follows by (iii) that

$$D^+V(t, x)_{(3.15.2)} \leqslant \alpha(t)V(t, x) + K_2\| F(t, x)\|. \tag{3.15.5}$$

Let $\epsilon > 0$ and $t_0 \in J$ be given. Because of condition (3.15.3), we have, if ϵ is small enough,

$$\lim_{t \to \infty} \exp \left[\frac{K_2 \epsilon}{K_1} (t - t_0) + \int_{t_0}^{t} \alpha(s)\, ds\right] = 0.$$

If we choose $\| x_0 \| < \delta_1$, where $K_2\delta_1 B < K_1\delta(\epsilon)$ and

$$B = \max_{t_0 \leqslant t < \infty} \exp\left[\frac{K_2\epsilon}{K_1}(t - t_0) + \int_{t_0}^{t} \alpha(s)\, ds\right],$$

then we get $\| x(t)\| < \delta(\epsilon)$, $t \geqslant t_0$. For, otherwise, there would exist a $t_1 > t_0$ with the property that

$$\| x(t)\| \leqslant \delta(\epsilon), \qquad t \in [t_0, t_1], \qquad \| x(t_1)\| = \delta(\epsilon).$$

Taking into account (3.15.4) and (3.15.5) and setting $m(t) = V(t, x(t))$, we obtain, by Theorem 1.4.1, the inequality

$$m(t) \leqslant m(t_0) \exp\left[\frac{K_2\epsilon}{K_1}(t - t_0) + \int_{t_0}^{t} \alpha(s)\, ds\right], \tag{3.15.6}$$

for $t \in [t_0, t_1]$, which, in turn, yields, at $t = t_1$,

$$\begin{aligned} K_1\delta(\epsilon) \leqslant m(t_1) \leqslant m(t_0) \exp\left[\frac{K_2\epsilon}{K_1}(t_1 - t_0) + \int_{t_0}^{t_1} \alpha(s)\, ds\right] \\ < K_2\delta_1 B \leqslant K_1\delta(\epsilon). \end{aligned}$$

This contradiction proves that, if $\| x_0 \| < \delta_1$, $\| x(t)\| < \delta(\epsilon)$, $t \geqslant t_0$, and so, the inequality (3.15.6) is true for all $t \geqslant t_0$. It is now easy to see that $\lim_{t\to\infty} x(t) = 0$, establishing the theorem.

THEOREM 3.15.4. In addition to the assumptions of Theorem 3.15.3, suppose that, for a constant $L > 0$,

$$\| f_x(t, x) - f_x(t, 0)\| \leqslant L\| x \|. \tag{3.15.7}$$

If $y(t) = y(t, t_0, x_0)$ is the solution of the variational system

$$y' = f_x(t, x(t))y, \qquad y(t_0) = x_0, \tag{3.15.8}$$

where $x(t) = x(t, t_0, x_0)$ is the solution of (3.15.2), $\| x_0 \|$ being sufficiently small, then $\lim_{t\to\infty} y(t) = 0$.

Proof. Let us first observe that

$$D^+V(t, y)_{(3.15.8)} \leqslant D^+V(t, y) + K_2\| f_x(t, x(t)) - f_x(t, 0)\| \, \| y \|.$$

If we now set $m(t) = V(t, y(t))$, we readily obtain, in view of (3.15.7) and condition (ii), the inequality

$$D^+m(t) \leqslant \alpha(t)m(t) + LK_2\| x(t)\| \, \| y(t)\|. \tag{3.15.9}$$

Since the hypotheses of Theorem 3.15.3 hold, if $\| x_0 \|$ is small enough, we have $\| x(t) \| < \epsilon$, $t \geqslant t_0$. Consequently, choosing $\| x_0 \|$ sufficiently small and using the relation $K_1 \| x \| \leqslant V(t, x)$, it follows that

$$D^+ m(t) \leqslant \left[\alpha(t) + L \frac{K_2 \epsilon}{K_1} \right] m(t),$$

which leads to the estimate

$$V(t, y(t)) \leqslant V(t_0, x_0) \exp \left[\frac{L K_2 \epsilon}{K_1} (t - t_0) + \int_{t_0}^{t} \alpha(s) \, ds \right]$$

for $t \geqslant t_0$. If ϵ is small enough, the condition (3.15.3) assures that $\lim_{t \to \infty} y(t) = 0$.

We shall next consider a theorem on the dependence of solutions on the initial values, which is useful in what follows.

THEOREM 3.15.5. Suppose that $f(t, x)$ is continuous on an open set D in $J \times R^n$ and that every solution of (3.15.2), $(t_0, x_0) \in D$ is continuable to $t = t_1 > 0$. Let E be the set of all the points consisting of the solution curves for $[t_0, t_1]$ starting from (t_0, x_0), and let E be contained in a compact set in D. Then, to each $\epsilon > 0$, there exists a $\delta > 0$ such that every solution $x^*(t, t_0^*, x_0^*)$, $t_0^* \in [t_0, t_1]$, of

$$x' = f(t, x) + g(t), \tag{3.15.10}$$

where $g \in C[J, R]$ and satisfies

$$\int_{t_0}^{t_1} g(s) \, ds < \delta, \tag{3.15.11}$$

passing through $p^* = (t_0^*, x_0^*)$ such that $d(p^*, E) \leqslant \delta$, exists on $[t_0^*, t_1]$ and obeys

$$\| x^*(t, t_0^*, x_0^*) - x(t, t_0, x_0) \| < \epsilon,$$

$x(t, t_0, x_0)$ being a solution of (3.15.2) contained in E, which may depend on $x^*(t, t_0^*, x_0^*)$.

Proof. Suppose that, for some $\epsilon > 0$, there is no δ such that it satisfies the condition in Theorem 3.15.5. We may assume that $\overline{U(E, \epsilon)} \subset D$, where $U(E, \epsilon) = [x: d(x, E) < \epsilon]$. Since $\overline{U(E, \epsilon)}$ is a compact set, there is a function $f^*(t, x)$ that is continuous and bounded on $(-\infty, \infty) \times R^n$ and is equal to $f(t, x)$ on $\overline{U(E, \epsilon)}$. A solution of (3.15.2) remaining in $\overline{U(E, \epsilon)}$ is a solution of

$$x' = f^*(t, x), \tag{3.15.12}$$

and the set of all the points consisting of the solution curves for $t \in [t_0, t_1]$ of (3.15.12) through (t_0, x_0) coincides with E. We may therefore assume that, for $\epsilon > 0$ and the equation

$$x' = f^*(t, x) + g(t), \tag{3.15.13}$$

the conclusion of the theorem is not verified. Every solution of (3.15.13) exists for all t. By hypothesis, there are a sequence of points $\{p_k = (t_k, x_k)\}$ and a sequence of functions $\{g_k(t)\}$ such that $d(p_k, E)$ tends to zero,

$$\int_{t_0}^{t_1} \|g_k(t)\| \, dt \to 0 \qquad \text{as} \quad k \to \infty,$$

and a solution $\phi_k(t)$, $t_k \leqslant t \leqslant t_1$, of

$$x' = f^*(t, x) + g_k(t)$$

through p_k such that there is no solution curve of (3.15.12) lying in E with the property that the distances of all the points on the arc of the former to the latter are smaller than ϵ. Since $\phi_k(t)$ is defined on $t_0 \leqslant t \leqslant t_1$,

$$\begin{aligned}\phi_k(t) &= x_k + \int_{t_k}^{t} f^*(s, \phi_k(s)) \, ds + \int_{t_k}^{t} g_k(s) \, ds \\ &= \phi_k(t_0) + \int_{t_0}^{t} f^*(s, \phi_k(s)) \, ds + \int_{t_0}^{t} g_k(s) \, ds, \end{aligned} \tag{3.15.14}$$

and thus $\{\phi_k(t)\}$ is uniformly bounded and equicontinuous on $[t_0, t_1]$. Hence, we can select a uniformly convergent subsequence. Denote its index by k again, and let $\phi(t)$ be its limit function. Because of (3.15.14), we have

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f^*(s, \phi(s)) \, ds,$$

and thus $\phi(t)$ is a solution of (3.15.12). If t_2 is a point of accumulation of $\{t_k\}$, then $(t_2, \phi(t_2)) \in E$. By $\phi(t)$ and a solution joining (t_0, x_0) and $(t_2, \phi(t_2))$, we have a solution $x = \phi^*(t)$ of (3.15.12) through (t_0, x_0). Therefore,

$$\phi^*(t) \subset E, \qquad \phi^*(t) = \phi(t), \qquad t \geqslant t_2 .$$

If k is sufficiently large and t_k is sufficiently close to t_2, the distance between $\phi_k(t)$ and $\phi(t)$ is smaller than ϵ, because $\phi_k(t)$ is uniformly convergent to $\phi(t)$. This contradicts our hypothesis. The theorem is thus proved.

Let us now consider a system of differential equations

$$x' = f(t, x) + R(t, x), \qquad x(t_0) = x_0, \tag{3.15.15}$$

where $f, R \in C[J \times E, R^n]$, E being an open set in R^n.

DEFINITION 3.15.1. A scalar function $v(x)$ defined for $x \in E$ is said to be *positive definite with respect to a set* $A \subset E$ if $v(x) = 0$ for $x \in A$ and, corresponding to every $\epsilon > 0$ and every compact set Q in E, there exists a positive number $\delta = \delta(Q, \epsilon)$ such that

$$v(x) \geqslant \delta \qquad \text{for} \quad x \in Q \cap S(A, \epsilon)^C,$$

where $S(A, \epsilon)^C$ denotes the complement of the set

$$S(A, \epsilon) = [x : d(x, A) < \epsilon].$$

DEFINITION 3.15.2. A solution $x(t)$ of (3.15.15) is said to approach a set A as $t \to \infty$ if, for each $\epsilon > 0$, there is a $T > 0$ with the property that, for all $t > T$, the points $x(t)$ are contained in $S(A, \epsilon)$.

THEOREM 3.15.6. Assume that the functions $f(t, x)$, $R(t, x)$, and $V(t, x)$ satisfy the following conditions for $(t, x) \in J \times E$:

(i) $f \in C[J \times E, R^n]$, and $f(t, x)$ is bounded for all $t \in J$ when x belongs to an arbitrary compact set in E.

(ii) $R \in C[J \times E, R^n]$, and, if $x(t)$ is continuous and bounded on $t_0 \leqslant t < \infty$, that is, $x(t) \subset Q$, Q being a compact set in E, then $R(t, x)$ satisfies the inequality

$$\int_{t_0}^{\infty} \| R(s, x(s))\| \, ds < \infty. \tag{3.15.16}$$

(iii) $V \in C[J \times E, R_+]$, $V(t, x)$ is locally Lipschitzian in x, and

$$\begin{aligned} D^+V(t, x) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + h[f(t, x) + R(t, x)]) - V(t, x)] \\ &\leqslant -C(x) + g(t, V(t, x)), \end{aligned} \tag{3.15.17}$$

where $C(x)$ is positive definite with respect to a closed set Ω in E, $g \in C[J \times R_+, R]$, and $g(t, u)$ is monotonic nondecreasing in u for each $t \in J$.

Then, if all the solutions of (3.15.15) and (3.2.3) are bounded, every solution of (3.15.15) approaches Ω as $t \to \infty$.

Proof. Let $x(t, t_0, x_0)$ be a solution of (3.15.15). By assumption, $x(t, t_0, x_0)$ is bounded, which implies that there is a compact set $\underset{\sim}{Q}$ in E such that

$$x(t, t_0, x_0) \in \underset{\sim}{Q}, \qquad t \geqslant t_0 .$$

If we suppose that this solution does not approach Ω as $t \to \infty$, then, for some $\epsilon > 0$, there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that

$$x(t_k, t_0, x_0) \in S(\Omega, \epsilon)^C \cap \underset{\sim}{Q}.$$

The assumption that $f(t, x)$ is bounded when $x \in \underset{\sim}{Q}$ assures that

$$\|f(t, x)\| \leqslant M$$

for some positive constant M. We may assume t_1 is sufficiently large so that, on intervals

$$t_k \leqslant t \leqslant t_k + \tfrac{1}{4}\epsilon/M, \tag{3.15.18}$$

we have, because of condition (3.15.16),

$$\int_{t_k}^{t_k+\frac{1}{4}\epsilon/M} \| R(s, x(s))\| \, ds < \tfrac{1}{4}\epsilon.$$

Thus, one gets, on the intervals (3.15.18), that

$$x(t, t_0, x_0) \in S(\Omega, \epsilon)^C \cap \underset{\sim}{Q}. \tag{3.15.19}$$

We may assume that these intervals are disjoint, if necessary, by taking a subsequence of $\{t_k\}$. The relation (3.15.17) and Theorem 3.1.3 give the inequality

$$V(t, x(t, t_0, x_0)) \leqslant -\int_{t_0}^{t} C[x(s, t_0, x_0)] \, ds + r(t, t_0, u_0) \tag{3.15.20}$$

for $t \geqslant t_0$, where $u_0 = V(t_0, x_0)$ and $r(t, t_0, u_0)$ is the maximal solution of (3.2.3). The positive definiteness of $C(x)$ with respect to Ω, together with the relation (3.15.19), shows the existence of a $\delta = \delta(\epsilon/2)$ such that

$$C[x(t, t_0, x_0)] \geqslant \delta, \qquad t_k \leqslant t \leqslant t_k + \frac{\epsilon}{4M}. \tag{3.15.21}$$

Moreover, from the boundedness of all solutions of (3.2.3), it follows that

$$r(t, t_0, u_0) < \beta, \qquad t \geqslant t_0 . \tag{3.15.22}$$

Thus, we arrive at the inequality

$$V\left(t_k + \frac{\epsilon}{4M}, x\left(t_k + \frac{\epsilon}{4M}, t_0, x_0\right)\right) < -\delta \frac{\epsilon}{4M} k + \beta,$$

in view of relations (3.15.20), (3.15.21), and (3.15.22). Since $V(t, x) \geqslant 0$, the foregoing relation leads to an absurdity as $k \to \infty$. This proves that every solution $x(t, t_0, x_0)$ approaches Ω as $t \to \infty$, and the proof is complete.

DEFINITION 3.15.3. A point $\omega \in R^n$ is said to be a cluster point of a solution $x(t)$ of (3.15.15) if there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that $x(t_k) \to \omega$ as $k \to \infty$. The set of cluster points is called the *positive limiting set* and is denoted by Γ^+.

If a solution $x(t, t_0, x_0)$ of (3.15.15) is bounded for $t \geqslant t_0$, then its positive limiting set Γ^+ is a nonempty, compact set, and $x(t, t_0, x_0) \to \Gamma^+$ as $t \to \infty$. Furthermore, if $x(t, t_0, x_0)$ is bounded for $t \geqslant t_0$ and if A contains the positive limiting set of $x(t, t_0, x_0)$, then $x(t, t_0, x_0) \to A$ as $t \to \infty$.

LEMMA 3.15.1. Let Ω be a closed set in E, that is, Ω is a closed set in the topology of E. Assume that a solution $x(t, t_0, x_0)$ is bounded and $x(t, t_0, x_0) \to \Omega$ as $t \to \infty$. Then the positive limiting set Γ^+ of $x(t, t_0, x_0)$ satisfies $\Gamma^+ \subset \Omega$.

Let $f(t, x)$ satisfy the following hypotheses:

(a) $f(t, x)$ tends to a function $H(x)$ for $x \in \Omega$ as $t \to \infty$, where Ω is a closed set in E, and, on any compact set in Ω, this convergence is uniform. Consequently, $H(x)$ is a continuous function on Ω.

(b) For each $\epsilon > 0$ and each $y \in \Omega$, there exist positive numbers $\delta(y)$ and $T(y)$ such that, if $\| x - y \| < \delta(y)$ and $t \geqslant T(y)$, we have

$$\| F(t, x) - F(t, y) \| < \epsilon.$$

If $t \in J$, then we can choose $\delta(y)$ so that (b) holds for all $t \geqslant 0$.

The following lemma can be proved in the same way as one proves the uniform continuity of a continuous function on a compact set. We merely state

LEMMA 3.15.2. If $y \in \Omega_1$, where Ω_1 is an arbitrary compact set in Ω, the δ and T of (b) are independent of y.

Now consider the differential system

$$x' = H(x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0, \tag{3.15.23}$$

where $H \in C[D, R^n]$, D being an open subset in R^n.

DEFINITION 3.15.4. A set $A \subset D$ is said to be a *semi-invariant* set of (3.15.23) if, for each point of A, there is at least one solution of (3.15.23) which remains in A for all future time.

The following theorem is stated in a special form that is convenient for applications. However, the proof can be modified to apply to a more general situation.

THEOREM 3.15.7. Assume that a solution $x(t, t_0, x_0)$ of (3.15.15) is bounded for $t \geqslant t_0$ and that it approaches a closed set Ω in E. Let $f(t, x)$ satisfy hypotheses (a) and (b). Suppose that $R(t, x)$ satisfies assumption (iii) of Theorem 3.15.6. Then, the positive limiting set Γ^+ of $x(t, t_0, x_0)$ is a semi-invariant set, contained in Ω, of Eq. (3.15.23)

Proof. By assumption, a solution $x(t) = x(t, t_0, x_0)$ of (3.15.15) is bounded in E, which implies that there is a compact set $\underset{\sim}{Q}$ such that

$$x(t, t_0, x_0) \in \underset{\sim}{Q}, \qquad t \geqslant t_0 .$$

By Lemma 3.15.1, we have

$$\Gamma^+ \subset \Omega \cap \underset{\sim}{Q} = \Omega_1 .$$

Since Ω_1 is a compact set in R^n, there exists a continuous, bounded function $H^*(x)$ on R^n such that $H^*(x) = H(x)$ on Ω_1. Consider the system

$$x' = H^*(x). \tag{3.15.24}$$

Let ω be a point of Γ^+. Then $\omega \in \Omega_1$, and there is a sequence $\{t_k\}$ such that

$$x(t_k) = x(t_k, t_0, x_0) \to \omega, \qquad t_k \to \infty \qquad \text{as} \quad k \to \infty. \tag{3.15.25}$$

Since (3.15.24) is an autonomous system, the behavior of the solutions of (3.15.24) through (t_k, ω) is the same as that of the solutions through $(0, \omega)$. For an arbitrary $\lambda > 0$, designate the interval $t_k \leqslant t \leqslant t_k + \lambda$ by J_k. The boundedness of $H^*(x)$ shows that all the solutions of (3.15.24) exist on J_k. Now consider the system

$$x' = H^*(x) + f(t, x(t)) - H^*(x(t)) + R(t, x(t)). \tag{3.15.26}$$

Clearly, $x = x(t)$ is a solution of (3.15.26) through $(t_k, x(t_k))$. As $x(t)$ is bounded, from condition (3.15.16), it follows that, if k is sufficiently large, that is, $k \geqslant k_1$, we have, for a given $\delta > 0$,

$$\int_{t_k}^{t_k+\lambda} \| R(s, x(s))\| \, ds < \tfrac{1}{2}\delta. \tag{3.15.27}$$

Since Ω_1 is compact, for every point $x(t)$ there is a point $y(t)$ satisfying

$$d(x(t), \Omega_1) = \| x(t) - y(t)\|.$$

By hypothesis (b) and Lemma 3.15.2, given $\delta/6\lambda$, there exist $\delta_1 > 0$ and $T > 0$ such that, if $y \in \Omega_1$, $\| x - y \| < \delta_1$, and $t \geqslant T$,

$$\|f(t, x) - f(t, y)\| < \delta/6\lambda.$$

On the other hand, since $x(t) \in \Omega_1$ as $t \to \infty$, for sufficiently large t, we have

$$x(t) \subset S(\Omega_1, \delta_1) \cap \underset{\sim}{Q}.$$

Hence, if t is sufficiently large, that is, $k \geqslant k_2$, it follows that

$$\|f(t, x(t)) - f(t, y(t))\| < \delta/6\lambda \qquad \text{on} \quad J_k. \tag{3.15.28}$$

Because of hypothesis (a), $f(t, x) \to H(x)$ as $t \to \infty$ for $x \in \Omega$, and this convergence is uniform for $x \in \Omega_1$, and, for sufficiently large t, the inequality

$$\|f(t, x) - H(x)\| < \delta/6\lambda$$

holds. If k is sufficiently large, that is, $k \geqslant k_3$, we therefore have

$$\|f(t, x(t)) - H(y(t))\| < \delta/6\lambda \qquad \text{on} \quad J_k. \tag{3.15.29}$$

Moreover, since $H^*(x)$ is continuous on $\underset{\sim}{Q}$, there is a $\delta_2 > 0$ such that, if $\| x - y \| < \delta_2$, $\| H^*(x) - H^*(y)\| < \delta/6\lambda$, and, if $y \in \Omega_1$, $H^*(y) = H(y)$. From this it follows that, if $y \in \Omega_1$ and $\| x - y \| < \delta_2$,

$$\| H^*(x) - H(y)\| < \delta/6\lambda.$$

Consequently, if $k \geqslant k_4$, we have, on J_k,

$$\| H^*(x(t)) - H(y(t))\| < \delta/6\lambda. \tag{3.15.30}$$

Thus, if $k \geqslant \max[k_1, k_2, k_3, k_4]$, from the relations (3.15.28), (3.15.29), and (3.15.30) and the inequality

$$\begin{aligned}\|f(t, x(t)) - H^*(x(t))\| &\leqslant \|f(t, x(t)) - f(t, y(t))\| \\ &\quad + \|f(t, x(t)) - H(y(t))\| + \| H(y(t)) - H^*(x(t))\|,\end{aligned}$$

we have

$$\|f(t, x(t)) - H^*(x(t))\| < \delta/2\lambda \qquad \text{on} \quad J_k. \tag{3.15.31}$$

The relations (3.15.27) and (3.15.31) yield that

$$\int_{t_k}^{t_k+\lambda} \|f(s, x(s)) - H^*(x(s)) + R(s, x(s))\| \, ds < \delta.$$

On the other hand, for sufficiently large k, (3.15.25) shows that

$$d((t_k, x(t_k)), (t_k, \omega)) \leqslant \delta.$$

Thus, by Theorem 3.15.6, there is a solution $\phi_k(t)$ of (3.15.24) through (t_k, ω) such that, for a given $\epsilon > 0$,

$$d(x(t), \phi_k(t)) < \epsilon, \qquad t \in J_k.$$

Since $\phi_k(t)$ is a solution (3.15.24) through (t_k, ω), we have

$$\phi_k(t) = \omega + \int_{t_k}^{t} H^*(\phi_k(s)) \, ds, \qquad t_k \leqslant t \leqslant t_k + \lambda,$$

and, if we denote $\phi_k(t + t_k)$, $0 \leqslant t \leqslant \lambda$, by $\phi_k(t)$ again, we deduce

$$\phi_k(t) = \omega + \int_0^t H^*(\phi_k(s)) \, ds, \qquad 0 \leqslant t \leqslant \lambda.$$

Therefore, for a sequence $\{\epsilon_k\}$, $\epsilon_k \to 0$ as $k \to \infty$, there exist solutions $\phi_k(t)$ of (3.15.24) satisfying

$$\begin{aligned} &\phi_k(t) = \omega + \int_0^t H^*(\phi_k(s)) \, ds, \qquad 0 \leqslant t \leqslant \lambda, \\ &\phi_k(t) \subset S(\Gamma^+, \epsilon_k). \end{aligned} \tag{3.15.32}$$

The sequence of functions $\{\phi_k(t)\}$ is uniformly bounded and equicontinuous, and hence we can select a subsequence that is uniformly convergent. Let $\phi(t)$ be its limit function. Then, it follows that

$$\phi(t) = \omega + \int_0^t H^*(\phi(s)) \, ds, \qquad 0 \leqslant t \leqslant \lambda,$$

and

$$\phi(t) \subset \Gamma^+, \qquad 0 \leqslant t \leqslant \lambda,$$

by (3.15.32) and the fact that $\epsilon_k \to 0$ as $k \to \infty$. Since $\Gamma^+ \subset \Omega_1$, this implies

$$H^*(\phi(t)) = H(\phi(t)),$$

and therefore

$$\phi(t) = \omega + \int_0^t H(\phi(s)) \, ds, \qquad 0 \leqslant t \leqslant \lambda.$$

This means that $\phi(t)$ is a solution of (3.15.23) through $(0, \omega)$, which remains in Γ^+. Because of the arbitrary nature of λ, one concludes that there is a solution of (3.15.23) defined for $t \geqslant 0$ starting from ω at $t = 0$ and remaining in Γ^+. Consequently, Γ^+ is a semi-invariant set of (3.15.23), and the proof is complete.

COROLLARY 3.15.1. If, for a solution $x(t)$ of (3.15.15) approaching Ω, $\lim_{t\to\infty} x(t) = 0$, then $H(x_0) = 0$.

The following theorem gives sufficient conditions for the asymptotic behavior of solutions of (3.15.15) whose proof follows by combining those of Theorems 3.15.6 and 3.15.7.

THEOREM 3.15.8. Let the hypotheses of Theorem 3.15.6 hold. Let $f(t, x)$ satisfy hypotheses (a) and (b). Then, all the solutions of (3.15.15) approach the largest semi-invariant set, contained in Ω, of Eq. (3.15.23).

3.16. Relative stability

The concept of relative stability is concerned with the following two differential systems:

$$\begin{aligned} x' &= f_1(t, x), \qquad x(t_0) = x_0, \\ y' &= f_2(t, y), \qquad y(t_0) = y_0, \end{aligned} \tag{3.16.1}$$

where $f_1, f_2 \in C[J \times R^n, R^n]$. Let $x(t) = x(t, t_0, x_0)$, $y(t) = y(t, t_0, y_0)$ be any two solutions of (3.16.1).

DEFINITION 3.16.1. The two differential systems (3.16.1) are said to be (R_1) *relatively equi-stable* if, for each $\epsilon > 0$ and $t_0 \in J$, there exists a $\delta = \delta(t_0, \epsilon)$ which is continuous in t_0 for each ϵ such that the inequality

$$\| x_0 - y_0 \| < \delta$$

implies

$$\| x(t) - y(t) \| < \epsilon, \qquad t \geqslant t_0.$$

Analogous to the definitions (S_2)–(S_{10}), we can formulate (R_2)–(R_{10}), following (R_1). If $f_1(t, y) \equiv f_2(t, y)$, then the notion (R_1) will be designated as *extreme equistability* of the product system (3.16.1). On the other hand, suppose that $f_2(t, y) \equiv 0$ and $y \in M$, where M is a nonempty subset in R^n. Let $d(x, M)$ denote the distance between a point x and the set M, defined by

$$d(x, M) = \inf[\| x - y \|, y \in M].$$

Since $d(x, M) \leqslant \| x - y \|$, for all $y \in M$, we can infer, as a special case, the stability with respect to a set M. If, furthermore, $M = \{0\}$ and $f_1(t, 0) \equiv 0$, the definitions (R_1)–(R_{10}) coincide with (S_1)–(S_{10}). Thus, the study of relative stability is important in itself.

THEOREM 3.16.1. Suppose that the following conditions hold:

(i) $V \in C[J \times R^n \times R^n, R_+]$, $V(t, x, x) \equiv 0$, and $V(t, x, y)$ is locally Lipschitzian in x and y.

(ii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and

$$D^+V(t, x, y)_{(3.16.1)} \leqslant g(t, V(t, x, y)), \qquad (t, x, y) \in J \times R^n \times R^n.$$

Then

(1) if the trivial solution of (3.2.3) is equistable and

$$b(\| x - y \|) \leqslant V(t, x, y), \qquad b \in \mathscr{K}, \tag{3.16.2}$$

the two systems (3.16.1) are relatively equistable;

(2) if the trivial solution of (3.2.3) is uniform stable and

$$b(\| x - y \|) \leqslant V(t, x, y) \leqslant a(\| x - y \|), \qquad a, b \in \mathscr{K}, \tag{3.16.3}$$

the two systems are relatively uniform stable;

(3) if (3.16.2.) holds, the equi-asymptotic stability of the trivial solution of (3.2.3) implies the relative equi-asymptotic stability of the two systems (3.16.1);

(4) if (3.16.3) holds, the uniform asymptotic stability of the trivial solution of (3.2.3) implies the relative uniform asymptotic stability of the two systems (3.16.1).

Proof. (1) Let $\epsilon > 0$, $t_0 \in J$ be given. By the equistability of the trivial solution of (3.2.3), we have, given $b(\epsilon) > 0$, $t_0 \in J$, that there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $u_0 \leqslant \delta$ implies

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0. \tag{3.16.4}$$

It follows from the continuity of the function $V(t, x, y)$ and the fact that $V(t, x, x) \equiv 0$ that it is possible to find $\delta_1 = \delta_1(t_0, \epsilon)$ satisfying

$$\| x_0 - y_0 \| \leqslant \delta_1, \qquad V(t_0, x_0, y_0) \leqslant \delta$$

at the same time. Defining

$$m(t) = V(t, x(t), y(t)),$$

and using assumptions (i) and (ii), we get the inequality

$$D^+m(t) \leqslant g(t, m(t)).$$

We choose $u_0 = V(t_0, x_0, y_0)$ and apply Theorem 1.4.1 to obtain

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0, \tag{3.16.5}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.2.3). From (3.16.2), (3.16.4), and (3.16.5), it follows that

$$b(\| x(t) - y(t)\|) < b(\epsilon), \qquad t \geqslant t_0,$$

Hence, $\| x_0 - y_0 \| < \delta_1$ implies $\| x(t) - y(t)\| < \epsilon$, $t \geqslant t_0$, proving (1).

(2) We choose $u_0 = a(\| x_0 - y_0 \|)$, and then it is enough to select $\delta_1 = a^{-1}(\delta)$. The proof proceeds as before. It is clear that δ_1 is independent of t_0, since δ does not depend on t_0.

(3) It follows from the hypotheses that, given $b(\epsilon) > 0$, $t_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that $u_0 \leqslant \delta_0$ implies that

$$u(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 + T.$$

We choose $\hat{\delta}_0 = \hat{\delta}_0(t_0)$ such that

$$\| x_0 - y_0 \| \leqslant \hat{\delta}_0, \qquad V(t_0, x_0, y_0) \leqslant \delta_0$$

hold simultaneously and conclude, as before, that

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0.$$

Consequently, it follows that

$$b(\| x(t) - y(t)\|) < b(\epsilon), \qquad t \geqslant t_0 + T.$$

Thus, whenever $\| x_0 - y_0 \| \leqslant \hat{\delta}_0$, we have

$$\| x(t) - y(t)\| < \epsilon, \qquad t \geqslant t_0 + T,$$

proving (R_3). Since, by (1), (R_1) holds, equi-asymptotic stability results.

(4) Choose $u_0 = a(\| x_0 - y_0 \|)$, and then select $\hat{\delta}_0 = a^{-1}(\delta_0)$. Clearly, $\hat{\delta}_0$ and T are independent of t_0. We proceed as in (3) to establish (R_6). The theorem is proved.

3.17. Stability with respect to a manifold

Let $w \in C[R^n, R^k]$, $k < n$, and let the set of points x satisfying the relation $w(x) = 0$ define an $(n - k)$ dimensional manifold. We define

$$\| w \| = \left[\sum_{i=1}^{k} w_i^2(x) \right]^{1/2}$$

and denote by $M_{(n-k)}(\epsilon)$ and $\bar{M}_{(n-k)}(\epsilon)$ the sets $[x \in R^n : \| w(x) \| < \epsilon]$ and $[x \in R^n : \| w(x) \| \leqslant \epsilon]$, respectively.

Definition 3.17.1. A set $A \subset R^n$ is said to be (positively) *self-invariant* if $x_0 \in A$ implies $x(t, t_0, x_0) \subset A$, $t \geqslant t_0$, where $x(t, t_0, x_0)$ is any solution of (3.2.1).

We shall assume that $M_{(n-k)}$ is a self-invariant set with respect to the system (3.2.1).

Definition 3.17.2. The self-invariant manifold $M_{(n-k)}$ is said to be (M_1) *equistable* if, for each $\epsilon > 0$ and $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that

$$x_0 \in \bar{M}_{(n-k)}(\delta)$$

implies

$$x(t, t_0, x_0) \subset M_{(n-k)}(\epsilon), \qquad t \geqslant t_0 .$$

Analogously, the definitions (M_2)–(M_8) may be understood, parallel to the definitions (S_2)–(S_8). Obviously, if $k = n$ and $w(x) = x$, these definitions (M_1)–(M_8) coincide with (S_1)–(S_8).

The following theorem gives sufficient conditions for the stability and asymptotic stability of the invariant manifold $M_{(n-k)}$, the proof of which is left to the reader as an easy exercise.

Theorem 3.17.1. Assume that there exist functions $V(t, x)$ and $g(t, u)$ satisfying the following conditions:

(i) $V \in C[J \times M_{(n-k)}(\rho), R_+]$, $V(t, x) = 0$ if $x \in M_{(n-k)}$, and $V(t, x)$ is locally Lipschitzian in x.

(ii) $g \in C[J \times R_+, R]$, and $g(t, 0) \equiv 0$.

(iii) $f \in C[J \times M_{(n-k)}(\rho), R^n]$, where $M_{(n-k)}$ is a self-invariant manifold with respect to the system (3.2.1), and

$$D^+ V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times M_{(n-k)}(\rho).$$

Then, the self-invariant manifold is

(1) equistable if the trivial solution of (3.2.3) is equistable and

$$b(\| w(x)\|) \leqslant V(t, x); \qquad b \in \mathscr{K}, \qquad (t, x) \in J \times M_{(n-k)}(\rho); \quad (3.17.1)$$

(2) uniformly stable if the trivial solution of (3.2.3) is uniformly stable and

$$b(\| w(x)\|) \leqslant V(t, x) \leqslant a(\| w(x)\|), \qquad (t, x) \in J \times M_{(n-k)}(\rho), \quad (3.17.2)$$

where, $a, b \in \mathscr{K}$;

(3) equi-asymptotically stable if the trivial solution of (3.2.3) is equi-asymptotically stable and (3.17.1) holds;

(4) uniformly asymptotically stable if the trivial solution of (3.2.3) is uniformly asymptotically stable and (3.17.2) is satisfied.

3.18. Almost periodic systems

We shall consider, in this section, uniqueness of solutions, existence of almost periodic solutions, and stability results allowing the initial time t_0 the perfect freedom of taking any value in the interval $(-\infty, \infty)$. However, the Lyapunov function that will be used is defined only for $t \geqslant 0$. In other words, we obtain results for any $t_0 \in (-\infty, \infty)$, although the conditions imposed in terms of Lyapunov function are only for $t \geqslant 0$. This definite advantage inherent in periodic or almost periodic systems is exhibited in what follows. Let us begin by a uniqueness result.

THEOREM 3.18.1. Assume that

(i) $f \in C[(-\infty, \infty) \times S_\rho, R^n]$, and $f(t, x)$ is almost periodic in t uniformly with respect to $x \in S$, S being any compact set in S_ρ;

(ii) $V \in C[J \times S_\rho \times S_\rho, R_+]$, $V(t, x, x) \equiv 0$, $V(t, x, y)$ is Lipschitzian in x and y for a constant $M = M(\rho) > 0$, and

$$b(\| x - y \|) \leqslant V(t, x, y), \qquad b \in \mathscr{K};$$

(iii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and, for $(t, x, y) \in J \times S_\rho \times S_\rho$,

$$D^+V(t, x, y) = \limsup_{h \to 0^+} \frac{1}{h} [V[t + h, x + hf(t, x), y + hf(t, y)] - V(t, x, y)]$$
$$\leqslant g(t, V(t, x, y));$$

(iv) the maximal solution of (3.2.3) through the point $(\tau_0, 0)$, $\tau_0 \geqslant 0$, is identically zero.

Then, the almost periodic system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \in (-\infty, \infty) \tag{3.18.1}$$

has at most one solution to the right of t_0.

Proof. Suppose that, for some (t_0, x_0), $t_0 \in (-\infty, \infty)$, $x_0 \in S_\rho$, there exist two solutions $x(t) = x(t, t_0, x_0)$, $y(t) = y(t, t_0, x_0)$. Then, at a certain $t_1 > t_0$, we have

$$\| x(t_1) - y(t_1) \| = \epsilon,$$

where, we may assume, $\epsilon < \rho$. For $t_0 \leqslant t \leqslant t_1$, there exists a constant $B < \rho$ such that

$$\| x(t) \| \leqslant B, \qquad \| y(t) \| \leqslant B.$$

By Lemma 1.3.1, given $\frac{1}{2}b(\epsilon)$ and a compact set $[\tau_0, T]$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$r(t, \tau_0, 0, \delta) \leqslant \tfrac{1}{2}b(\epsilon), \qquad t \in [\tau_0, T], \tag{3.18.2}$$

where $r(t, \tau_0, 0, \delta)$ is the maximal solution of

$$u' = g(t, u) + \delta, \qquad u(\tau_0) = 0. \tag{3.18.3}$$

Let θ be a $\delta(\epsilon)/2M$-translation number for $f(t, x)$ such that $t_0 + \theta \geqslant 0$, that is,

$$\| f(t + \theta, x) - f(t, x) \| < \delta(\epsilon)/2M, \tag{3.18.4}$$

provided $x \in$ a compact set $S \subset S_\rho$. Consider the function $m(t) = V(t + \theta, x(t), y(t))$ for $t \in [t_0, t_1]$. For small $h > 0$, we have, using Lipschitz condition on $V(t, x, y)$,

$$\begin{aligned} m(t + h) \leqslant Mh \Big[&\| f(t, x(t)) - f(t + \theta, x(t)) \| \\ &+ \| f(t, y(t)) - f(t + \theta, y(t)) \| + \frac{\| \epsilon_1(h) \|}{h} + \frac{\| \epsilon_2(h) \|}{h} \Big] \\ &+ V[t + \theta + h, x(t) + hf(t + \theta, x(t)), y(t) + hf(t + \theta, y(t))], \end{aligned}$$

where $\epsilon_1(h)/h$, $\epsilon_2(h)/h \to 0$ as $h \to 0$. It then follows, because of assumption (iii) and relation (3.18.4), that

$$D^+ m(t) \leqslant g(t + \theta, m(t)) + \delta, \qquad t \in [t_0, t_1].$$

Defining $\tau_0 = t_0 + \theta$, we get, on the strength of Theorem 1.4.1, the inequality

$$m(t) \leqslant r(t + \theta, \tau_0, 0, \delta), \qquad t \in [t_0, t_1], \tag{3.18.5}$$

where $r(t + \theta, \tau_0, 0, \delta)$ is the maximal solution of (3.18.3). At $t = t_1$, we obtain, using relations (3.18.2) and (3.18.5), the estimate

$$m(t_1) \leqslant r(t_1 + \theta, \tau_0, 0, \delta) \leqslant \tfrac{1}{2}b(\epsilon),$$

which is a contradiction to the fact that

$$m(t_1) \geqslant b(\epsilon).$$

Hence, it follows that the system (3.18.1) has at most one solution to the right of t_0.

COROLLARY 3.18.1. The function $g(t, u) \equiv 0$ is admissible in Theorem 3.18.1.

As observed earlier, in the stability results that follow, we allow $t_0 \in (-\infty, \infty)$. Then, the corresponding notions will be designated as *perfect stability* concepts to distinguish them from the previous stability definitions. We need the following notions with respect to the scalar differential equation (3.2.3).

DEFINITION 3.18.1. The trivial solution of (3.2.3) is said to be *strongly equistable* if, given any $\epsilon > 0$, $\tau_0 \in J$, and any compact interval $K = [\tau_0, t_1]$, there exist an $\eta = \eta(\epsilon) > 0$ and a positive function $\delta = \delta(\tau_0, \epsilon)$ that is continuous in τ_0 for each ϵ such that, if $u_0 \leqslant \delta$,

$$u(t, \tau_0, u_0, \eta) < \epsilon, \qquad t \in [\tau_0, t_1],$$

where $u(t, \tau_0, u_0, \eta)$ is any solution of

$$u' = g(t, u) + \eta, \qquad u(\tau_0) = u_0 \geqslant 0. \tag{3.18.6}$$

If, in addition, δ is independent of τ_0, it is *strongly uniformly stable.*

DEFINITION 3.18.2. The trivial solution of (3.2.3) is said to be *strongly equi-asymptotically stable* if it is strongly equistable and if, for any $\epsilon > 0$, $\tau_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(\tau_0)$, $\eta = \eta(\epsilon)$, and $T = T(\tau_0, \epsilon)$ such that

$$u(t, \tau_0, u_0, \eta) < \epsilon, \qquad t \geqslant \tau_0 + T,$$

provided

$$u_0 \leqslant \delta_0,$$

where $u(t, \tau_0, u_0, \eta)$ is any solution of (3.18.6). If the numbers δ, δ_0, and T are independent of τ_0, the trivial solution is said to be *strongly uniformly asymptotically stable.*

THEOREM 3.18.2. Assume that

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is Lipschitzian in x for a constant $L = L(\rho) > 0$, and $V(t, x)$ is positive definite;

(ii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times S_\rho;$$

(iii) $f \in C[(-\infty, \infty) \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $f(t, x)$ is almost periodic in t uniformly with respect to $x \in S$, S being any compact set in S_ρ.

Then, the strong equistability of the trivial solution of (3.2.3) implies that the null solution of (3.18.1) is perfectly equistable.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in (-\infty, \infty)$ be given. Since $V(t, x)$ is positive definite, there exists a function $b \in \mathscr{K}$ such that

$$b(\| x \|) \leqslant V(t, x), \qquad (t, x) \in J \times S_\rho. \tag{3.18.7}$$

Assume that the trivial solution of (3.2.3) is strongly equistable. Then, given $b(\epsilon) > 0$, $\tau_0 \in J$, and any compact interval $K = [\tau_0, t_1]$, there exist an $\eta = \eta(\epsilon) > 0$ and a $\delta = \delta(\tau_0, \epsilon) > 0$ such that

$$u(t, \tau_0, u_0, \eta) < b(\epsilon), \qquad t \in [\tau_0, t_1], \tag{3.18.8}$$

provided $u_0 \leqslant \delta$, where $u(t, \tau_0, u_0, \eta)$ is any solution of (3.18.6). Choose $L\delta_1 = \delta$ and $u_0 = L \| x_0 \|$, L being the Lipschitz constant for $V(t, x)$. This choice implies that $u_0 \leqslant \delta$ and $\| x_0 \| \leqslant \delta_1$ are satisfied at the same time.

Suppose now that there exists a solution $x(t) = x(t, t_0, x_0)$, with $\| x_0 \| \leqslant \delta_1$ and $t_0 \in (-\infty, \infty)$ such that, for some $t_2 > t_0$, we have

$$\| x(t) \| \leqslant \epsilon < \rho, \qquad t_0 \leqslant t \leqslant t_2, \qquad \| x(t_2) \| = \epsilon. \tag{3.18.9}$$

Let θ be an η/L-translation number for $f(t, x)$ such that $t_0 + \theta \geqslant 0$, that is,

$$\| f(t + \theta, x) - f(t, x) \| < \eta/L, \qquad t \in (-\infty, \infty), \tag{3.18.10}$$

if $x \in S$, any compact set in S_ρ. We consider the function $m(t) = V(t + \theta, x(t))$, $t \in [t_0, t_2]$. If $h > 0$ is small, we obtain, using assumption

(ii), the Lipschitzian character of $V(t, x)$ and the relations (3.18.9) and (3.18.10),

$$\begin{aligned} D^+m(t) &\leqslant g(t+\theta, m(t)) + L\|f(t+\theta, x(t)) - f(t, x(t))\| \\ &\leqslant g(t+\theta, m(t)) + \eta, \qquad t \in [t_0, t_2]. \end{aligned}$$

Define $\tau_0 = t_0 + \theta$ and $t_1 = t_2 + \theta$. An application of Theorem 1.4.1 yields that

$$m(t) \leqslant r(t+\theta, \tau_0, u_0, \eta), \qquad t \in [t_0, t_2],$$

where $r(t+\theta, \tau_0, u_0, \eta)$ is the maximal solution of (3.18.6). At $t = t_2$, there results an absurdity

$$b(\epsilon) \leqslant V(t_2+\theta, x(t_2)) \leqslant r(t_2+\theta, \tau_0, u_0, \eta) < b(\epsilon),$$

because of relations (3.18.7), (3.18.8), and (3.18.9). This proves the perfect equistability of the trivial solution of (3.18.1).

COROLLARY 3.18.2. Under the assumptions of Theorem 3.18.2, the strong uniform stability of the trivial solution of (3.2.3) assures the perfect uniform stability of the solution $x = 0$ of (3.18.1). In particular, the function $g(t, u) \equiv 0$ is admissible.

THEOREM 3.18.3. Suppose the trivial solution of (3.2.3) is strongly equi-asymptotically stable and that assumptions (i), (ii), and (iii) of Theorem 3.18.2 hold. Then, the perfect equi-asymptotic stability of the trivial solution of (3.18.1) follows.

Proof. Since, by Theorem 3.18.2, the trivial solution of (3.18.1) is perfectly equistable, it remains to be proved that it is perfectly quasi-equi-asymptotically stable. For this purpose, let $0 < \epsilon < \rho$ and $t_0 \in (-\infty, \infty)$. Then, given $b(\epsilon) > 0$ and $\tau_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(\tau_0)$, $\eta = \eta(\epsilon)$, and $T = T(\tau_0, \epsilon)$ such that

$$u(t, \tau_0, u_0, \eta) < b(\epsilon), \qquad t \geqslant \tau_0 + T, \tag{3.18.11}$$

whenever $u_0 \leqslant \delta$. Choose $u_0 = L\|x_0\|$ and $L\delta_0 = \delta_0$. Let $\delta_0^* = \min[\delta_0, \tilde{\delta}_0]$, where $\tilde{\delta}_0 = \delta(\tau_0, \rho)$. Thus, if $\|x_0\| \leqslant \delta_0^*$, it follows that $\|x(t)\| \leqslant B < \rho$ for some B. As before, let θ be an η/L-translation number so that (3.18.10) is satisfied. Then, by defining $m(t) = V(t+\theta, x(t))$, where $x(t)$ is any solution of (3.18.1) such that $\|x_0\| \leqslant \delta_0^*$, we obtain

$$D^+m(t) \leqslant g(t+\theta, m(t)) + \eta,$$

which implies, by Theorem 1.4.1, the inequality

$$m(t) \leqslant r(t + \theta, \tau_0, u_0, \eta), \qquad t \geqslant t_0,$$

choosing $\tau_0 = t_0 + \theta$. It then follows that

$$\begin{aligned} b(\| x(t)\|) \leqslant V(t + \theta, x(t)) &\leqslant r(t + \theta, \tau_0, u_0, \eta) \\ &< b(\epsilon), \qquad t \geqslant t_0 + T, \end{aligned}$$

in view of (3.18.11). We thus have

$$\| x(t)\| < \epsilon, \qquad t \geqslant t_0 + T,$$

provided $\| x_0 \| \leqslant \delta_0^*$. The proof is therefore complete.

COROLLARY 3.18.3. Under the assumptions of Theorem 3.18.2, the strong uniform asymptotic stability of the trivial solution of (3.2.3) implies the perfect uniform asymptotic stability of the null solution of (3.18.1). In particular, the function $g(t, u) = -\alpha u$, $\alpha > 0$, is admissible.

Suppose that the function $f(t, x)$ is not almost periodic in t and that $f \in C[J \times S_\rho, R^n]$. Then, from the strong stability notions of the scalar differential equation (3.2.3), we may infer the strong stability of the system (3.18.1), which we now define.

DEFINITION 3.18.3. The trivial solution of (3.18.1) is said to be *strongly equistable* if, for any $\epsilon > 0$, $t_0 \in J$, and any compact interval $[t_0, t_1]$, there exist an $\eta = \eta(\epsilon) > 0$ and a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that, if $\| x_0 \| \leqslant \delta$,

$$\| x(t, t_0, x_0, \eta)\| < \epsilon, \qquad t \in [t_0, t_1],$$

where $x(t, t_0, x_0, \eta)$ is an η-approximate solution of (3.18.1) on $[t_0, t_1]$. If, in addition, δ is independent of t_0, the trivial solution is said to be *strongly uniform stable.*

The notion of strong asymptotic stability may be defined similar to Definition 3.18.2.

We may now state the following

THEOREM 3.18.4. Suppose that assumptions (i) and (ii) of Theorem 3.18.2 hold. Let $f \in C[J \times S_\rho, R^n]$ and $f(t, 0) \equiv 0$. Then, one of the strong stability definitions of the trivial solution of (3.2.3) yields the corresponding one of the strong stability notions of the trivial solution of (3.18.1).

Proof. We shall indicate the proof corresponding to the statement that the strong equistability of $u = 0$ of (3.2.3) implies the strong equistability of the trivial solution of (3.18.1). We follow the proof of Theorem 3.18.2 and choose δ_1, η_1 such that $L\delta_1 = \delta$ and $L\eta_1 = \eta$. We can, then, claim that, with the numbers δ_1, η_1 so chosen, the trivial solution of (3.18.1) is strongly equistable. Supposing the contrary and proceeding as in Theorem 3.18.2, we obtain, in the present case, the inequality

$$D^+m(t) \leqslant g(t, m(t)) + \eta,$$

where $m(t) = V(t, x(t, t_0, x_0, \eta))$. With these changes, we can mimic the rest of the arguments to prove the stated result.

Regarding the existence of almost periodic solutions for the system (3.18.1), we have the following

THEOREM 3.18.5. Assume that

(i) $f \in C[(-\infty, \infty) \times S_\rho, R^n]$, and $f(t, x)$ is almost periodic in t uniformly with respect to $x \in S$, S being any compact set in S_ρ;

(ii) $V \in C[J \times S_\rho \times S_\rho, R_+]$, $V(t, x, y)$ is Lipschitzian in x and y for a constant $L = L(\rho) > 0$, and, for $(t, x, y) \in J \times S_\rho \times S_\rho$,

$$b(\| x - y \|) \leqslant V(t, x, y) \leqslant a(\| x - y \|), \qquad a, b \in \mathscr{K};$$

(iii) $D^+V(t, x, y) \leqslant g(t, V(t, x, y)), \qquad t \in J, x, y \in S_\rho$, where $g \in C[J \times R_+, R]$;

(iv) there exists a solution $x(t) = x(t, t_0, x_0)$ of (3.18.1) such that

$$\| x(t)\| \leqslant B, \qquad t \geqslant t_0, \qquad t_0 \in (-\infty, \infty), \qquad B < \rho;$$

(v) given $b(\epsilon) > 0$, $\alpha > 0$, and $\tau_0 \in J$, there exist positive numbers $\eta = \eta(\epsilon)$, $T = T(\epsilon, \alpha)$ such that, if $u_0 \leqslant \alpha$ and $\tau \geqslant \tau_0 + T$,

$$u(\tau, \tau_0, u_0, \eta) < b(\epsilon) \tag{3.18.12}$$

where $u(\tau, \tau_0, u_0, \eta)$ is any solution of (3.18.6). Then, the almost periodic system (3.18.1) admits a bounded almost periodic solution, with a bounded B.

Proof. We can prove this theorem following the proof of Theorem 2.15.3. Hence, we shall indicate it briefly. Let $x(t) = x(t, t_0, x_0)$, $t_0 \in (-\infty, \infty)$ be the bounded solution such that $\| x(t)\| \leqslant B$, $t \geqslant t_0$. Let $\{\tau_k\}$ be any sequence of real numbers such that $\tau_k \to \infty$ as $k \to \infty$ and

$f(t + \tau_k, x) - f(t, x) \to 0$ as $k \to \infty$, uniformly for $t \in (-\infty, \infty)$ and $x \in S$, S being any compact set in S_ρ. Let β be any number, and let U be any compact subset of $[\beta, \infty)$. Let $0 < \epsilon < \rho$ be given. Choose $\alpha = a(2B)$. Then, let η and T be the numbers given in assumption (v), for this choice of α. Let $k_0 = k_0(\beta)$ be the smallest value of k such that $\beta + \tau_{k_0} \geqslant t_0 + T$. Choose an integer $\eta_0 = \eta_0(\epsilon, \beta) \geqslant k_0$ so large that, for $k_2 \geqslant k_1 \geqslant \eta_0$,

$$\|f(t + \tau_{k_1}, x) - f(t + \tau_{k_2}, x)\| \leqslant \eta/3L, \tag{3.18.13}$$

for all $t \in (-\infty, \infty)$, $x \in S$. Let θ be an $\eta/3L$-translation number for $f(t, x)$ such that $t_0 + \theta \geqslant 0$, that is,

$$\|f(t + \theta, x) - f(t, x)\| \leqslant \eta/3L, \tag{3.18.14}$$

for $t \in (-\infty, \infty)$ and $x \in S$. Consider the function

$$m(t) = V(t + \theta, x(t), x(t_1)), \qquad t \geqslant t_0,$$

where $t_1 = t + \tau_{k_2} - \tau_{k_1}$. Then,

$$\begin{aligned}
D^+m(t) &\leqslant \limsup_{h \to 0^+} \frac{1}{h} [V(t + \theta + h, x(t + h), x(t_1 + h)) \\
&\quad - V(t + \theta + h, x(t) + hf(t + \theta, x(t)), x(t_1) + hf(t + \theta, x(t_1)))] \\
&\quad + \limsup_{h \to 0^+} \frac{1}{h} [V(t + \theta + h, x(t) + hf(t + \theta, x(t)), x(t_1) \\
&\quad + hf(t + \theta, x(t_1))) - V(t + \theta, x(t), x(t_1))] \\
&\leqslant L[\|f(t, x(t)) - f(t + \theta, x(t))\| + \|f(t_1, x(t_1)) \\
&\quad - f(t + \theta, x(t_1))\|] + D^+V(t + \theta, x(t), x(t_1)) \\
&\leqslant L[\|f(t, x(t)) - f(t + \theta, x(t))\| + \|f(t_1, x(t_1)) \\
&\quad - f(t_1 + \theta, x(t_1))\| + \|f(t_1 + \theta, x(t_1)) - f(t + \theta, x(t_1))\|] \\
&\quad + g(t + \theta, m(t)),
\end{aligned}$$

using the Lipschitzian character of $V(t, x, y)$ and assumption (iii).

Since $t + \tau_{k_1} \geqslant t_0$, for $t \in U$, we obtain, by virtue of the relations (3.18.13) and (3.18.14), the inequality

$$D^+m(t + \tau_{k_1}) \leqslant g(t + \tau_{k_1} + \theta, m(t + \tau_{k_1})) + \eta, \qquad t \in U,$$

which, by Theorem 1.4.1, yields

$$m(t + \tau_{k_1}) \leqslant r(t + \tau_{k_1} + \theta, t_0 + \theta, u_0, \eta), \qquad t + \tau_{k_1} \geqslant t_0,$$

provided $m(t_0) = u_0$, where $r(\tau, \tau_0, u_0, \eta)$ is the maximal solution of (3.18.6). But, for all $t \in U$, $t + \tau_{k_1} \geqslant t_0 + T$. Hence, identifying $\tau = t + \tau_{k_1} + \theta$, $\tau_0 = t_0 + \theta$, we get

$$m(t + \tau_{k_1}) < b(\epsilon), \qquad t \in U,$$

according to relation (3.18.12). Consequently, for all $t \in U$, we have

$$\| x(t + \tau_{k_1}) - x(t + \tau_{k_2})\| < \epsilon, \qquad k_2 \geqslant k_1 \geqslant \eta_0 .$$

This proves the existence of a function $w(t)$ defined on $[\beta, \infty)$ and bounded by B. Since β is arbitrary, $w(t)$ is defined for $t \in (-\infty, \infty)$, and we have

$$x(t + \tau_{k_1}) - w(t) \to 0 \qquad \text{as} \quad k_1 \to \infty,$$

uniformly on all compact subsets of $(-\infty, \infty)$.

Using the same arguments as in Theorem 2.15.3, it is easy to show that $w(t)$ is differentiable and satisfies (3.18.1).

To show that $w(t)$ is almost periodic, it is sufficient to show that, for any sequence $\{\tau_k\}$ for which $\{f(t + \tau_k, x)\}$ converges uniformly for $t \in (-\infty, \infty)$, $x \in S$, the sequence $\{w(t + \tau_k)\}$ converges uniformly for $t \in (-\infty, \infty)$, where τ_k tends to a finite number or infinity. We may assume that τ_k approaches either $-\infty$ or ∞.

Assume now that $\tau_k \to \infty$ as $k \to \infty$. For any $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon) > 0$ such that, if $k_2 \geqslant k_1 \geqslant n_0$, (3.18.13) holds. We choose $k_1 \geqslant n_0$ so large that $\tau_{k_1} \geqslant T$. For each $t \in (-\infty, \infty)$, let θ be an $\eta/3L$-translation number such that $t + \theta \geqslant 0$, that is, (3.18.14) holds. For $t \leqslant s \leqslant t + \tau_{k_1}$, we consider the function

$$m(s) = V(s + \theta, w(s), w(s_1)),$$

where $s_1 = s + \tau_{k_2} - \tau_{k_1}$. Then we obtain, as before,

$$\begin{aligned} D^+m(s) \leqslant g(s + \theta, m(s)) + L[&\|f(s, w(s)) - f(s + \theta, w(s))\| \\ &+ \|f(\sigma + \tau_{k_2}, w(s_1)) - f(\sigma + \tau_{k_1}, w(s_1))\| \\ &+ \|f(\sigma + \tau_{k_1}, w(s_1)) - f(\sigma + \tau_{k_1} + \theta, w(s_1))\|], \end{aligned}$$

where $\sigma = s - \tau_{k_1}$. This implies, using (3.18.13) and (3.18.14), that

$$D^+m(s) \leqslant g(s + \theta, m(s)) + \eta, \qquad t \leqslant s \leqslant t + \tau_{k_1} .$$

By Theorem 1.4.1, it follows that

$$m(\xi) \leqslant r(\xi + \theta, t + \theta, u_0, \eta), \qquad \xi \geqslant t,$$

if $m(t) = u_0$. As previously, choosing $\alpha = a(2B)$, we get, by relation (3.18.12),

$$r(\xi + \theta, t + \theta, u_0, \eta) < b(\epsilon), \qquad \xi \geqslant t + T.$$

If we set, therefore, $\xi = t + \tau_{k_1}$, there results

$$m(t + \tau_{k_1}) < b(\epsilon), \qquad t \in (-\infty, \infty).$$

Accordingly, it follows that

$$\| w(t + \tau_{k_1}) - w(t + \tau_{k_2}) \| < \epsilon, \qquad k_2 \geqslant k_1 \geqslant n_0,$$

for all $t \in (-\infty, \infty)$. Thus, the sequence $\{w(t + \tau_k)\}$ is convergent uniformly on $(-\infty, \infty)$.

In the case when $\tau_k \to -\infty$, we can prove in the same manner that $\{w(t + \tau_k)\}$ is also convergent uniformly on $(-\infty, \infty)$. The proof is therefore complete.

Corollary 3.18.4. Let hypotheses (i), (ii), (iii), and (iv) of Theorem 3.18.5 hold, and let the trivial solution of (3.2.3) be strongly uniformly asymptotically stable. Then, the system (3.18.1) admits an almost periodic solution that is uniformly asymptotically stable. In particular, the function $g(t, u) = -\alpha u$, $\alpha > 0$, is admissible.

3.19. Uniqueness and estimates

It is naturally possible to give very general conditions for the uniqueness and the growth of solutions by employing Lyapunov functions. Let us begin with a uniqueness result of Perron type.

Theorem 3.19.1. Assume that

(i) the function $g(t, u)$ is continuous for $t_0 \leqslant t \leqslant t_0 + a$, $u \geqslant 0$, and, for every t_1, $t_0 < t_1 < t_0 + a$, $u(t) \equiv 0$ is the only differentiable function on $t_0 \leqslant t < t_1$, which satisfies

$$u' = g(t, u), \qquad u(t_0) = 0, \tag{3.19.1}$$

for $t_0 < t < t_1$;

(ii) $f \in C[R_0, R^n]$, where

$$R_0 : t_0 \leqslant t \leqslant t_0 + a, \qquad \| x - x_0 \| \leqslant b;$$

(iii) $V \in C[R_0, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ is positive definite, continuously differentiable on R_0, and, for $(t, x), (t, y) \in R_0$,

$$V'(t, x-y) = \frac{\partial v(t, x-y)}{\partial t} + \frac{\partial v(t, x-y)}{\partial x} \cdot [f(t, x) - f(t, y)]$$

$$\leqslant g(t, V(t, x-y)). \tag{3.19.2}$$

Then, the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{3.19.3}$$

admits a unique solution on $t_0 \leqslant t \leqslant t_0 + a$.

Proof. Let us suppose that there are two solutions $x(t)$, $y(t)$ of the system (3.19.3) on $t_0 \leqslant t \leqslant t_0 + a$. Consider the function

$$m(t) = V(t, x(t) - y(t)).$$

We have, in view of (3.19.2) and the continuous differentiability of $V(t, x)$, the inequality

$$m'(t) \leqslant g(t, m(t)).$$

Also, $m(t_0) = 0$. For any t_1 such that $t_0 < t_1 < t_0 + a$, we obtain, by Theorem 1.4.1, the estimate

$$m(t) \leqslant r(t), \qquad t_0 \leqslant t < t_1,$$

where $r(t)$ is the maximal solution of (3.19.1). Assumption (i), together with the positive definiteness of V, assures that $x(t) \equiv y(t)$, $t_0 \leqslant t < t_1$. The proof is therefore complete.

We can state a uniqueness result analogous to Kamke's theorem in terms of Lyapunov functions as follows.

THEOREM 3.19.2. Assume that

(i) the function $g(t, u)$ is continuous for $t_0 < t \leqslant t_0 + a$, $u \geqslant 0$, and, for every t_1, $t_0 < t_1 < t_0 + a$, $u(t) \equiv 0$ is the only function differentiable on $t_0 < t < t_1$ and continuous on $t_0 \leqslant t < t_1$, for which

$$u'_+(t_0) = \lim_{t \to t_0^+} \frac{u(t) - u(t_0)}{t - t_0} \quad \text{exists,}$$

$$u'(t) = g(t, u(t)), \qquad t_0 < t < t_1,$$

and

$$u(t_0) = u'_+(t_0) = 0;$$

(ii) hypotheses (ii) and (iii) of Theorem 3.19.1 hold except that the condition (3.19.2) is satisfied only for $(t, x), (t, y) \in R_0$ and $t \neq t_0$.

Then, the conclusion of Theorem 3.19.1 remains true.

Proof. Define the function

$$g_f(t, u) = \sup_{V(t,x-y)=u} V'(t, x-y)$$

for $t_0 \leqslant t \leqslant t_0 + a$ and $u \geqslant 0$. Since $f(t, x)$, $\partial V/\partial t$, $\partial V/\partial x$ are all continuous on R_0, it follows that $g_f(t, u)$ is continuous on $t_0 \leqslant t \leqslant t_0+a$ and $u \geqslant 0$. Following the proof of Theorem 2.2.2, it is now easy to establish this theorem. We leave the details.

Much the same way, we can state and prove a uniqueness theorem analogous to Theorem 2.2.4.

The next two theorems are concerned with estimating the difference between a solution and an approximate solution.

THEOREM 3.19.3. Suppose that

(i) $g \in C[J \times R_+, R]$, $u, \delta \in C[J, R_+]$, and

$$D_-u(t) > g(t, u(t)) + M(t)\delta(t), \qquad t > t_0;$$

(ii) $V \in C[J \times R^n, R_+]$, $V(t, x)$ satisfies the Lipschitz condition in x for a function $M(t)$, where $M \in C[J, R_+]$, and

$$\begin{aligned} D^+V(t, x-y) &= \limsup_{h\to 0^+} \frac{1}{h} [V(t+h, x-y+h[f(t, x) - f(t, y)]) \\ &\quad - V(t, x-y)] \\ &\leqslant g(t, V(t, x-y)] \end{aligned} \tag{3.19.4}$$

for $t \in J$, $x, y \in \Omega$, where

$$\Omega = [x, y \in R^n : V(t, x-y) = u(t), t > t_0];$$

(iii) $f \in C[J \times R^n, R^n]$, $x(t)$ is a δ-approximate solution, and $y(t)$ a solution of (3.19.3), defined for $t \geqslant t_0$.

Then, $V(t_0, x_0 - y_0) < u(t_0)$ implies

$$V(t, x(t) - y(t)) < u(t), \qquad t \geqslant t_0.$$

Proof. Consider the function $m(t) = V(t, x(t) - y(t))$. If, for a $t = t_1$, $x(t_1), y(t_1) \in \Omega$, then we obtain, using assumption (ii), the differential inequality

$$D^+m(t_1) \leqslant g(t_1, m(t_1)) + M(t_1)\delta(t_1),$$

which, by Lemma 1.2.3, implies

$$D_-m(t_1) \leqslant g(t_1, m(t_1)) + M(t_1)\delta(t_1).$$

The desired result now follows on account of Theorem 1.2.2.

THEOREM 3.19.4. Assume that $g \in C[J \times R_+, R]$ and that $r(t)$ is the maximal solution of

$$u' = g(t, u) + M(t)\delta(t), \qquad u(t_0) = u_0 \geqslant 0,$$

defined for $t \geqslant t_0$. Suppose further that assumptions (ii) and (iii) of Theorem 3.19.3 hold except that the condition (3.19.4) is satisfied for $t \in J$, $x, y \in \Omega^*$, where, for a certain $\epsilon_0 > 0$,

$$\Omega^* = [x, y \in R^n : r(t) < V(t, x - y) < r(t) + \epsilon_0, t \geqslant t_0].$$

Then, $V(t_0, x_0 - y_0) = u_0$ implies

$$V(t, x(t) - y(t)) \leqslant r(t), \qquad t \geqslant t_0.$$

The proof of this theorem can be constructed following the arguments of Theorem 1.4.2.

3.20. Continuous dependence and the method of averaging

Consider the differential system

$$x' = f(t, x, y), \tag{3.20.1}$$

where $f \in C[J \times S_\rho \times R^m, R^n]$. Assume that, for every $y_0 \in R^m$, there exists a solution $x_0(t)$ of the system

$$x' = f(t, x, y_0), \tag{3.20.2}$$

defined for $t \geqslant 0$.

LEMMA 3.20.1. Suppose that

(i) $V \in C[J \times R^n, R_+]$, $V(t, 0) \equiv 0$, and $V(t, x)$ is positive definite and satisfies the Lipschitz condition in x for a constant $M > 0$;

(ii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and, for any step function $v(t)$ on J with values in S_ρ and for every $t \in J$, $x \in S_\rho$,

$$\begin{aligned} D^+U(t, x, y_0) &= \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, F(t, v(t), y_0) - x \\ &\quad + h\{f(t, v(t), y_0) - f(t, x, y_0)\}) - V(t, F(t, v(t), y_0) - x)] \\ &\leqslant g(t, V(t, v(t) - x)), \end{aligned} \tag{3.20.3}$$

where

$$F(t, v(t), y) = v(0) + \int_0^t f(s, v(s), y)\, ds. \tag{3.20.4}$$

Then, given any compact interval $[0, T_0]$ contained in J and any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that, for every step function $v(t)$ in $[0, T_0]$, with $v(0) = x_0(0)$ and $\| v(t) - x_0(t)\| < \delta$ in $[0, T_0]$, there follows

$$\left\| \int_0^t [f(s, v(s), y_0) - f(s, x_0(s), y_0)]\, ds \right\| < \epsilon, \tag{3.20.5}$$

for every $t \in [0, T_0]$, $x_0(t)$ being any solution of (3.20.2).

Proof. Since $V(t, x)$ is positive definite and $V(t, 0) \equiv 0$, given any $\epsilon > 0$, there exists a $\mu = \mu(\epsilon) > 0$ such that

$$V(t, x) < \mu \qquad \text{implies} \quad \| x \| < \epsilon.$$

Moreover, by the assumptions on $g(t, u)$, there exists a $\gamma = \gamma(\epsilon) > 0$ such that

$$| g(t, u)| < \mu/T_0,$$

whenever $t \in [0, T_0]$ and $0 \leqslant u \leqslant \gamma$. Let $v(t)$ be a step function in $[0, T_0]$, with values in S_ρ such that $v(0) = x_0(0)$ and $\| v(t) - x_0(t)\| < \delta$ in $[0, T_0]$, where $\delta = \gamma/M$, M being the Lipschitz constant for $V(t, x)$. It then follows that

$$| g(t, V(t, v(t) - x_0(t)))| < \mu/T_0, \qquad t \in [0, T_0]. \tag{3.20.6}$$

Hence, considering the function

$$m(t) = V(t, F(t, v(t), y_0) - x_0(t)),$$

we obtain, in view of the relations (3.20.3) and (3.20.6) and the Lipschitzian character of $V(t, x)$,

$$D^+m(t) < \mu/T_0, \qquad t \in [0, T_0].$$

This implies that, for every $t \in [0, T_0]$,

$$V\left(t, \int_0^t [f(s, v(s), y_0) - f(s, x_0(s), y_0)]\, ds\right)$$
$$\equiv V(t, F(t, v(t), y_0) - x_0(t))$$
$$< \mu,$$

which proves the assertion (3.20.5) of the lemma.

Assume that, for each $t \in J$ and $x \in S_\rho$,

$$\lim_{y \to y_0} \int_0^t f(s, x, y)\, ds = \int_0^t f(s, x, y_0)\, ds. \tag{3.20.7}$$

It then follows that, given any compact interval $[0, T_0]$ and any step function $v(t)$ in $[0, T_0]$ with values in S_ρ,

$$\lim_{y \to y_0} \int_0^t f(s, v(s), y)\, ds = \int_0^t f(s, v(s), y_0)\, ds$$

uniformly in $[0, T_0]$. Hence, if the assumptions of Lemma 3.20.1 hold, there exists, for every $\epsilon > 0$, a constant $\delta = \delta(\epsilon) > 0$ such that, whenever $v(t)$ is a step function in $[0, T_0]$ with $v(0) = x_0(0)$ and $\| v(t) - x_0(t)\| < \delta$ in $[0, T_0]$, there is a neighborhood $\Gamma = \Gamma(\epsilon) \subset R^m$ of y_0 for which $y \in \Gamma$ implies

$$\left\| \int_0^t [f(s, v(s), y) - f(s, x_0(s), y_0)]\, ds \right\| < \epsilon, \qquad t \in [0, T_0].$$

In view of (3.20.4), this means that, for each $y \in \Gamma$,

$$\| F(t, v(t), y) - x_0(t)\| < \epsilon, \qquad t \in [0, T_0].$$

This fact will be used prominently in the following

THEOREM 3.20.1. Assume that

(i) $V \in C[J \times R^n, R_+]$, $V(t, 0) \equiv 0$, and $V(t, x)$ is positive definite and satisfies the Lipschitz condition in x for a constant $M > 0$;

(ii) $g \in C[J \times R_+, R]$, $g(t, 0) \equiv 0$, and $r(t) \equiv 0$ is the maximal solution of

$$u' = g(t, u), \tag{3.20.8}$$

passing through $(0, 0)$;

(iii) for any step function $v(t)$ on J, with values in S_ρ, and for every $t \in J$, $x \in S_\rho$, $y \in R^m$,

$$D^+U(t, x, y) \leqslant g(t, V(t, v(t) - x)); \tag{3.20.9}$$

(iv) the relation (3.20.7) holds.

Then, given any compact interval $[0, T_0] \subset J$ and any $\epsilon > 0$, there exists a neighborhood $\Gamma(\epsilon)$ of y_0 such that, for every $y \in \Gamma$, (3.20.1) admits a unique solution $x(t)$ with $x(0) = x_0(0)$, which is defined in $[0, T_0]$ and satisfies

$$\| x(t) - x_0(t)\| < \epsilon, \qquad t \in [0, T_0]. \tag{3.20.10}$$

Proof. The assumptions on V and g, together with (3.20.9), imply, on the basis of Theorem 3.19.1, that there exists a unique solution $x(t)$ of (3.20.1) with $x(0) = x_0(0)$, which is defined in some interval $J(y) = [0, T(y)] \subset J$.

From hypothesis (ii) and Lemma 1.3.1, we deduce that, given any compact interval $[0, T_0] \subset J$ and any $\mu > 0$, there is an $\eta = \eta(\mu) > 0$ such that the maximal solution $r(t, 0, 0, \eta)$ of

$$u' = g(t, u) + \tfrac{1}{2}\eta \tag{3.20.11}$$

exists for $t \in [0, T_0]$ and satisfies

$$r(t, 0, 0, \eta) < \mu, \qquad t \in [0, T_0].$$

Let $\epsilon > 0$ and $[0, T_0]$ be an arbitrary compact interval. Since $V(t, x)$ is positive definite on $J \times R^n$, we can find a $\mu = \mu(\epsilon) > 0$ such that, whenever $V(t, x) < \mu$, we have $\| x \| < \frac{1}{2}\epsilon$. Let $\eta(\epsilon) > 0$ be the constant referred to previously. Choose a constant $\alpha > 2M\epsilon$. By the continuity of g on $J \times R_+$, there exists a $\delta(\epsilon) > 0$ such that

$$| g(t, u_1) - g(t, u_2)| < \tfrac{1}{2}\eta$$

for $t \in [0, T_0]$, $u_1, u_2 \in [0, \alpha]$, and $| u_1 - u_2 | < \delta(\epsilon)$. For every $y \in R^m$ and every step function v in J with values in S_ρ, we have, for every $t \in J$ and $x \in R^n$,

$$\begin{aligned} &| V(t, v(t) - x) - V(t, F(t, v(t), y) - x)| \\ &\qquad \leqslant M[\| v(t) - x_0(t)\| + \| F(t, v(t), y) - x_0(t)\|], \end{aligned}$$

where $F(t, v(t), y)$ is defined as in (3.20.4). Hence, as observed earlier, we can select a positive constant $\beta(\epsilon) < \epsilon$ and a step function v in $[0, T_0]$

with $v(0) = x_0(0)$ and $\| v(t) - x_0(t)\| < \beta$ in $[0, T_0]$ such that there is a neighborhood $\Gamma(\epsilon)$ of y_0 for which $y \in \Gamma(\epsilon)$ implies

$$\| F(t, v(t), y) - x_0(t)\| < \tfrac{1}{2}\epsilon, \qquad t \in [0, T_0], \tag{3.20.12}$$

and

$$| V(t, v(t) - x) - V(t, F(t, v(t), y) - x)| < \delta,$$

for every $t \in [0, T_0]$ and $x \in R^n$.

Let us now take some $y \in \Gamma(\epsilon)$, and let us consider the unique solution $x(t)$ of (3.20.1), with $x(0) = x_0(0)$, which exists on some interval $J(y) = [0, T(y)]$ contained in J. Defining for $t \in J(y) \cap [0, T_0]$,

$$m(t) = V(t, F(t, v(t), y) - x(t)).$$

We deduce from (3.20.9) that

$$D^+m(t) \leqslant g(t, V(t, v(t) - x(t))).$$

Hence, for all those $t \in J(y) \cap [0, T_0]$ for which

$$\max[\| v(t) - x(t)\|, \| F(t, v(t), y) - x(t)\|] \leqslant \alpha/M, \tag{3.20.13}$$

there follows

$$D^+m(t) \leqslant g(t, m(t)) + \tfrac{1}{2}\eta.$$

This implies, by Theorem 1.4.1, that

$$m(t) \leqslant r(t, 0, 0, \eta), \qquad t \in J(y) \cap [0, T_0],$$

where $r(t, 0, 0, \eta)$ is the maximal solution of (3.20.11) through (0, 0). Since $r(t, 0, 0, \eta) < \mu$ for every $t \in [0, T_0]$, we infer that $m(t) < \mu$ as long as (3.20.13) holds, and therefore

$$\| F(t, v(t), y) - x(t)\| < \frac{\epsilon}{2} < \frac{\alpha}{4M}\cdot$$

Thus, using (3.20.12), we obtain, for sufficiently small $t \in J(y) \cap [0, T_0]$,

$$\| x(t) - x_0(t)\| < \epsilon, \tag{3.20.14}$$

and so $\| v(t) - x(t)\| < 2\epsilon < \alpha/M$, for these values of t. Consequently, (3.20.14) holds for every $t \in J(y) \cap [0, T_0]$. All that remains to be shown is that $[0, T_0] \subset J(y)$.

Suppose the contrary, and let $J^* = [0, T^*)$, with $T^* < T_0$, be the

maximal interval in which $x(t)$ exists. Since $x_0(t)$ is bounded in $[0, T_0]$, (3.20.14) shows that $x(t)$ is bounded in J^*. It follows that

$$\|f(t, x(t), y)\| \leqslant N \quad \text{in} \quad J^*$$

for some $N = N(y) > 0$. Hence, $x(t)$ has a limit C as $t \in J^*$ tends to T^*, and, by continuity,

$$\| C - x_0(T^*)\| \leqslant \epsilon$$

in view of (3.20.14). Since there exists a constant $\gamma > 0$ such that, for each $t \in [0, T_0]$, the open ball B_t in R^n of center $x_0(t)$ and radius γ is contained in S_ρ, we shall have $C \in S_\rho$ provided $\epsilon > 0$ was chosen sufficiently small. Therefore, $x(t)$ can be continued as a solution of (3.20.1) to the compact interval $[0, T^*]$, which contradicts the definition of J^*. This completes the proof.

If the assumption (3.20.7) is replaced by the stronger requirement

$$\lim_{y \to y_0} f(t, x, y) = f(t, x, y_0)$$

uniformly in $J \times S_\rho$, then we can prove the conclusion (3.20.10) without the use of approximating step functions. This we state in the form of a corollary, observing that it is a generalization of Theorem 2.5.2.

COROLLARY 3.20.1. Let assumptions (i) and (ii) of Theorem 3.20.1 hold. Suppose that, for each $t \in J$, $x_1, x_2 \in S_\rho$, and $y \in R^m$,

$$\limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x_1 - x_2 + h\{f(t, x_1, y) - f(t, x_2, y)\}) - V(t, x_1 - x_2)]$$
$$\leqslant g(t, V(t, x_1 - x_2)). \tag{3.20.15}$$

Then, the conclusion of Theorem 3.20.1 is true.

We next consider the problem of continuity of solutions with respect to initial values. We first prove the following

LEMMA 3.20.2. Suppose that

(i) $V \in C[J \times R^n, R_+]$, and $V(t, x)$ satisfies a Lipschitz condition in x locally;

(ii) $f \in C[J \times R^n, R^n]$, and

$$G(t, m) = \max_{V(t,x-x_0) \leqslant m} D^+V(t, x - x_0),$$

where

$$D^+V(t, x - x_0) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x - x_0 + hf(t, x))) - V(t, x - x_0)];$$

(iii) $r^*(t, t_0, 0)$ is the maximal solution of

$$u' = G(t, u), \qquad u(t_0) = 0$$

existing for $t \geqslant t_0$.
Then, if $x(t)$ is any solution of

$$x' = f(t, x), \qquad x(t_0) = x_0 \tag{3.20.16}$$

existing for $t \geqslant t_0$, we have

$$V(t, x(t) - x_0) \leqslant r^*(t, t_0, 0), \qquad t \geqslant t_0.$$

Proof. Define $m(t) = V(t, x(t) - x_0)$. Then, it is readily seen that

$$\begin{aligned} D^+m(t) &\leqslant D^+V(t, x(t) - x_0) \\ &\leqslant \max_{V(t,x-x_0)\leqslant m(t)} D^+V(t, x - x_0) \\ &= G(t, m(t)), \end{aligned}$$

which implies, by Theorem 1.4.1, that

$$m(t) \leqslant r^*(t, t_0, 0), \qquad t \geqslant t_0.$$

This proves the lemma.

We now state the following theorem on continuity of solutions $x(t, t_0, x_0)$ with respect to initial values, whose proof may be constructed by combining the arguments of Theorems 2.5.1, 3.1.4, 3.19.1, and 3.20.1.

THEOREM 3.20.2. Assume that

(i) $V \in C[J \times R^n, R_+]$, $V(t, 0) \equiv 0$, and $V(t, x)$ is positive definite, mildly unbounded, and satisfies a Lipschitz condition in x locally;

(ii) $g \in C[J \times R_+, R]$, and $u(t) \equiv 0$ is the unique solution of

$$u' = g(t, u) \tag{3.20.17}$$

passing through $(t_0, 0)$;

(iii) $f \in C[J \times R^n, R^n]$, and, for $(t, x), (t, y) \in J \times R^n$,

$$D^+V(t, x - y) \leqslant g(t, V(t, x - y)).$$

Then, if the solutions $u(t, t_0, u_0)$ of (3.20.17) through every point (t_0, u_0) exist for $t \geqslant t_0$ and are continuous with respect to (t_0, u_0), the solutions $x(t, t_0, x_0)$ of (3.20.16) exist for $t \geqslant t_0$ and are unique and continuous with respect to initial values (t_0, x_0).

3.21. Notes

A result of the type given in Theorem 3.1.1 is due to Conti [1]. Corollary 3.1.2 is new. Theorem 3.1.2 is adopted from Lakshmikantham [6, 10]. Theorem 3.1.3 is also new and is useful in certain applications. For the result contained in Theorem 3.1.4, see Brauer [3], Conti [1], Lakshmikantham [6, 10], and Strauss [1]. See also Wintner [1]. Instead of $D^+V(t, x)$ given in (3.1.2), it is more general to consider $D^-V(t, x)$. The proofs do not require any changes (see Corduneanu [11] and Yoshizawa [2]).

Section 3.2 introduces various definitions (see Antosiewicz [4], Hahn [1, 3, 4], LaSalle and Lefschetz [1], Lakshmikantham [6], Massera [4], and Yoshizawa [16]). For relationships between various kinds of stability and boundedness, see Antosiewicz [4], Massera [4], and Yoshizawa [16].

The results of Sect. 3.3 are adapted from the work of Antosiewicz [4, 6], Brauer [3, 8], Carduneanu [11], and Lakshmikantham [6, 10]. Theorem 3.3.5 is taken from Hahn [1, 3, 4], whereas Theorem 3.3.6 is due to Corduneanu [11]. Theorems 3.3.7 and 3.3.8 are adapted from Halanay [2]. Most of the results of Sect. 3.4 are based on the work of Antosiewicz [4, 6], Brauer [3, 8], Corduneanu [11], and Lakshmikantham [6, 10]. See also Hahn [1, 3, 4], Persidskiĭ [4], and Yoshizawa [16].

Section 3.5 deals with the results concerning the preservation of stability properties of unperturbed systems under certain classes of perturbations. Theorem 3.5.1 is due to Corduneanu [8], whereas Theorem 3.5.2 is new. See also Corduneanu [11].

Theorems 3.6.1 and 3.6.2 are based on the work of Yoshizawa [2, 16]. The proof of Theorem 3.6.3 is new. Theorems 3.6.4–3.6.8 are due to Lakshmikantham and Leela [2]. See also Corduneanu [13]. Theorem 3.6.9 is due to Massera [4]. The proof in the text is taken from Halanay [2]. The condition $\int_t^{t+u} L(s)\, ds \leqslant Ku$, $u \geqslant 0$, is not more general than $L(t) \leqslant K$. Theorem 3.6.10 is due to Corduneanu [11], which is more useful than Theorem 3.6.9, while considering the stability of perturbed systems. See also Yoshizawa [16].

Theorems 3.7.1 and 3.7.2 are taken from Halanay [2]. The short proofs given in the text are new. Theorem 3.7.3 is due to Strauss and

Yorke [1], whereas Theorem 3.7.4 is new. Theorem 3.7.5 is based on the work of Hale [2]. For Theorems 3.7.6 and 3.7.7, see Halanay [2]. Theorem 3.7.8 is due to Corduneanu [15]. See also Halanay [2], Krasovskii [14], and Malkin [8]. For transformation of time in the problem of stability by the first approximation, see Bylov [1].

The results on total stability given in Sec 3.8 are adapted from Halanay [2]. The notion of integral stability is introduced by Vrkoc [2]. For an equivalent notion, see Hayashy [1]. The results of Sect. 3.9 are based on Halanay [2].

Section 3.10 consists of results adapted from the work of Strauss [3]; for a generalization, see Hahn [3]. The results of Sect. 3.11 are due to Corduneanu [16]. See also Halanay [2]. Section 3.12 contains the work of Lakshmikantham [11]. Results of Sect. 3.13 are based on the work of Antosiewicz [4], Lakshmikantham [6, 10] and Yoshizawa [2, 16]. Theorem 3.13.11 is new.

The concept of eventual stability is due to LaSalle and Rath [1]. For a different version of this concept, see Lakshmikantham and Leela [1]. The results of Sect. 3.14 are based on the work of Wexler [1] and Yoshizawa [12, 15].

Theorems 3.15.1 and 3.15.2 are new, whereas Theorems 3.15.3 and 3.15.4 are adopted from Brauer [10, 12]. The rest of the results of Sect. 3.15 are due to Yoshizawa [10, 16]. Section 3.16 consists of the results due to Lakshmikantham [6], whereas the contents of Sect. 3.17 are due to Bhatia and Lakshmikantham [1]. See also Ling [1].

Section 3.18 contains results due to Lakshmikantham and Leela [3]. See also Deysach and Sell [1], Hale [2], Miller [1–3], Sell [4], Seifert [3, 5, 6], and Yoshizawa [16].

Uniqueness theorems 3.19.1 and 3.19.2 are based on the work of Brauer and Sternberg [1] and Olech [4]. Theorems 3.19.3 and 3.19.4 are new. Lemma 3.20.1 and Theorem 3.20.1 are due to Antosiewicz [7], whereas Lemma 3.20.2 and Theorem 3.20.2 are new.

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Chapter 4

4.0. Introduction

As we have seen, using a single Lyapunov function, it was possible to study a variety of problems in a unified way. It is natural to ask whether it might be more advantageous, in some situations, to use several Lyapunov functions. The answer is positive, and this approach leads to a more flexible mechanism. Moreover, each function can satisfy less rigid requirements. In this chapter, we attempt to obtain criteria for stability, instability, boundedness of solutions, and existence of stationary points, in terms of several Lyapunov functions.

4.1. Main comparison theorem

Let us consider the differential system

$$x' = f(t, x), \qquad x(t_0) = x_0, \qquad t_0 \geqslant 0. \tag{4.1.1}$$

Let $V \in C[J \times S_\rho, R_+^N]$. We define the vector function

$$D^+V(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)] \tag{4.1.2}$$

for $(t, x) \in J \times S_\rho$. The following theorem is an extension to systems of the corresponding theorem 3.1.1 and plays an important role whenever we use vector Lyapunov functions.

THEOREM 4.1.1. Let $V \in C[J \times S_\rho, R_+^N]$ and $V(t, x)$ be locally Lipschitzian in x. Assume that the vector function $D^+V(t, x)$ defined by (4.1.2) satisfies the inequality

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times S_\rho, \tag{4.1.3}$$

where $g \in C[J \times R_+^N, R^N]$, and the vector function $g(t, u)$ is quasi-monotone nondecreasing in u, for each fixed $t \in J$. Let $r(t, t_0, u_0)$ be the maximal solution of the differential system

$$u' = g(t, u), \qquad u(t_0) = u_0 \geqslant 0, \qquad t_0 \geqslant 0, \tag{4.1.4}$$

existing to the right of t_0. If $x(t) = x(t, t_0, x_0)$ is any solution of (4.1.1) such that

$$V(t_0, x_0) \leqslant u_0, \tag{4.1.5}$$

then, as far as $x(t)$ exists to the right of t_0, we have

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0). \tag{4.1.6}$$

Proof. Let $x(t, t_0, x_0)$ be any solution of (4.1.1) such that $V(t_0, x_0) \leqslant u_0$. Define the vector function $m(t)$ by

$$m(t) = V(t, x(t, t_0, x_0)).$$

Then, using the hypothesis that $V(t, x)$ satisfies Lipschitz's condition in x, we obtain, for small positive h, the inequality

$$\begin{aligned} m(t+h) - m(t) \leqslant K \| x(t+h) - x(t) - hf(t, x(t)) \| \\ + V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)), \end{aligned}$$

where K is the local Lipschitz constant. This, together with (4.1.1) and (4.1.3), implies the inequality

$$D^+ m(t) \leqslant g(t, m(t)).$$

Moreover, $m(t_0) \leqslant u_0$. Hence, by Corollary 1.7.1, we have

$$m(t) \leqslant r(t, t_0, u_0)$$

as far as $x(t)$ exists to the right of t_0, proving the desired relation (4.1.6).

We can now state a global existence theorem analogous to Theorem 3.1.4.

THEOREM 4.1.2. Assume that $V \in C[J \times R^n, R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, and $\sum_{i=1}^N V_i(t, x)$ is mildly unbounded. Suppose that $g \in C[J \times R_+^N, R^N]$, $g(t, u)$ is quasi-monotone nondecreasing in u for each fixed $t \in J$, and $r(t, t_0, u_0)$ is the maximal solution of (4.1.4) existing for $t \geqslant t_0$. If $f \in C[J \times R^n, R^n]$ and

$$D^+ V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times R^n,$$

then every solution $x(t) = x(t, t_0, x_0)$ of (4.1.1) exists in the future, and (4.1.5) implies (4.1.6) for all $t \geqslant t_0$.

By repeating the arguments used in the proof of Theorem 3.1.4, with appropriate changes, this theorem can be established.

On the basis of Corollary 1.7.1 and the remark that follows, we can prove the following:

THEOREM 4.1.3. Let $V \in C[J \times S_\rho, R_+^N]$ and $V(t, x)$ be locally Lipschitizan in x. Suppose that $g_1, g_2 \in C[J \times R_+^N, R^N]$, $g_1(t, u)$, $g_2(t, u)$ possess quasi-monotone nondecreasing property in u for each $t \in J$, and, for $(t, x) \in J \times S_\rho$,

$$g_1(t, V(t, x)) \leqslant D^+V(t, x) \leqslant g_2(t, V(t, x)).$$

Let $r(t, t_0, u_0)$, $\rho(t, t_0, v_0)$ be the maximal, minimal solutions of

$$u' = g_2(t, u), \qquad u(t_0) = u_0,$$
$$v' = g_1(t, v), \qquad v(t_0) = v_0,$$

respectively, such that

$$v_0 \leqslant V(t_0, x_0) \leqslant u_0.$$

Then, as far as $x(t) = x(t, t_0, x_0)$ exists to the right of t_0, we have

$$\rho(t, t_0, v_0) \leqslant V(t, x(t)) \leqslant r(t, t_0, u_0),$$

where $x(t)$ is any solution of (4.1.1).

4.2. Asymptotic stability

An approach that is extremely fruitful in proving asymptotic stability is to modify Lyapunov's original theorem without demanding $D^+V(t, x)$ to be negative definite. As we have seen, Theorem 3.15.8 is a very general result of this nature, although it covers a particular situation of the function $f(t, x)$. The theorem that follows takes care of the general case of $f(t, x)$ and requires two Lyapunov functions.

THEOREM 4.2.1. Suppose that the following conditions hold:

(i) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $f(t, x)$ is bounded on $J \times S_\rho$.

(ii) $V_1 \in C[J \times S_\rho, R_+]$, $V_1(t, x)$ is positive definite, decrescent, locally Lipschitzian in x, and

$$D^+V_1(t, x) \leqslant w(x) \leqslant 0, \qquad (t, x) \in J \times S_\rho,$$

where $w(x)$ is continuous for $x \in S_\rho$.

(iii) $V_2 \in C[J \times S_\rho, R_+]$, and $V_2(t, x)$ is bounded on $J \times S_\rho$ and is locally Lipschitzian in x. Furthermore, given any number α, $0 < \alpha < \rho$, there exist positive numbers $\xi = \xi(\alpha) > 0$, $\eta = \eta(\alpha) > 0$, $\eta < \alpha$, such that

$$D^+V_2(t, x) > \xi$$

for $\alpha < \| x \| < \rho$ and $d(x, E) < \eta$, $t \geqslant 0$, where

$$E = [x \in S_\rho : w(x) = 0]$$

and $d(x, E)$ is the distance between the point x and the set E.

Then, the trivial solution of (4.1.1) is uniformly asymptotically stable.

Proof. Let $\epsilon > 0$ and $t_0 \in J$ be given. Since $V_1(t, x)$ is positive definite and decrescent, there exist functions $a, b \in \mathscr{K}$ such that

$$b(\| x \|) \leqslant V_1(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times S_\rho. \tag{4.2.1}$$

We choose $\delta = \delta(\epsilon)$ so that

$$b(\epsilon) > a(\delta). \tag{4.2.2}$$

Then, arguing as in the first part of the proof of Theorem 3.4.9, we can conclude that the trivial solution of (4.1.1) is uniformly stable.

Let us now fix $\epsilon = \rho$ and define $\delta_0 = \delta(\rho)$. Let $0 < \epsilon < \rho$, $t_0 \in J$, and $\delta = \delta(\epsilon)$ be the same δ obtained in (4.2.2) for uniform stability. Assume that $\| x_0 \| < \delta_0$. To prove uniform asymptotic stability of the solution $x = 0$, it is enough to show that there exists a $T = T(\epsilon)$ such that, for some $t^* \in [t_0, t_0 + T]$, we have

$$\| x(t^*, t_0, x_0) \| < \delta.$$

This we achieve in a number of stages:

(1) If $d[x(t_1), x(t_2)] > r > 0$, $t_2 > t_1$, then

$$r \leqslant Mn^{1/2}(t_2 - t_1), \tag{4.2.3}$$

where $\|f(t, x)\| \leqslant M$, $(t, x) \in J \times S_\rho$. For, consider

$$
\begin{aligned}
| x_i(t_1) - x_i(t_2)| &\leqslant \int_{t_1}^{t_2} | x_i'(s)| \, ds \leqslant \int_{t_1}^{t_2} | f_i(s, x(s))| \, ds \\
&\leqslant M(t_2 - t_1) \qquad (i = 1, 2, \ldots, n),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
r &< d[x(t_1), x(t_2)] \\
&= \{[x_1(t_1) - x_1(t_2)]^2 + \cdots + [x_n(t_1) - x_n(t_2)]^2\}^{1/2} \\
&\leqslant Mn^{1/2}(t_2 - t_1).
\end{aligned}
$$

(2) By assumption (iii), given $\delta = \delta(\epsilon)$, $0 < \delta < \rho$, there exist $\xi = \xi(\epsilon)$, $\eta = \eta(\epsilon)$, $\eta < \delta$ such that

$$D^+V(t, x) > \xi, \qquad \delta < \| x \| < \rho, \qquad d(x, E) < \eta, \qquad t \geqslant 0.$$

Let us consider the set

$$U = [x \in S_\rho : \delta < \| x \| < \rho, d(x, E) < \eta],$$

and let

$$\sup_{\substack{\|x\| < \rho \\ t \geqslant 0}} V_2(t, x) = L.$$

Assume that, at $t = t_1$, $x(t_1) = x(t_1, t_0, x_0) \in U$. Then, for $t > t_1$, we have, letting $m(t) = V_2(t, x(t))$,

$$D^+m(t) \geqslant D^+V_2(t, x(t)) > \xi,$$

because of condition (iii) and the fact that $V_2(t, x)$ satisfies a Lipschitz condition in x locally. Thus,

$$m(t) - m(t_1) = \int_{t_1}^{t} D^+m(s) \, ds,$$

and hence

$$
\begin{aligned}
m(t) + m(t_1) &\geqslant \int_{t_1}^{t} D^+m(s) \, ds \geqslant \int_{t_1}^{t} D^+V_2(s, x(s)) \, ds \\
&> \xi(t - t_1)
\end{aligned}
$$

as long as $x(t)$ remains in U. This inequality can simultaneously be realized with $m(t) \leqslant L$ only if

$$t < t_1 + 2L/\xi.$$

It therefore follows that there exists a t_2, $t_1 < t_2 \leqslant t_1 + 2L/\xi$ such that $x(t_2)$ is on the boundary of the set U. In other words, $x(t)$ cannot stay permanently in the set U.

(3) Consider the sequence $\{t_k\}$ such that

$$t_k = t_0 + k\frac{2L}{\xi} \qquad (k = 0, 1, 2,\ldots).$$

Set $n(t) = V_1(t, x(t))$. Then, by assumption (ii), we have

$$D^+ n(t) \leqslant D^+ V_1(t, x(t)) \leqslant 0.$$

We let

$$\lambda = \inf[| w(x)|, \qquad \delta < \| x \| < \rho, \qquad d(x, E) \geqslant \eta/2],$$

and

$$\lambda_1 = \frac{\lambda\eta}{2Mn^{1/2}}.$$

Suppose that $x(t)$ is such that, for $t_k \leqslant t \leqslant t_{k+2}$, $\delta < \| x(t)\| < \rho$. If, for $t_k \leqslant t \leqslant t_{k+1}$, we have $\delta < \| x(t)\| < \rho$ and $d(x, E) \geqslant \frac{1}{2}\eta$, then, using assumption (ii) together with the definition of the set E, we obtain

$$\begin{aligned} n(t_{k+2}) - n(t_k) &= \int_{t_k}^{t_{k+2}} D^+ n(s)\, ds \\ &\leqslant \int_{t_k}^{t_{k+2}} D^+ V_1(s, x(s))\, ds \\ &\leqslant \int_{t_k}^{t_{k+1}} D^+ V_1(s, x(s))\, ds + \int_{t_{k+1}}^{t_{k+2}} D^+ V_1(s, x(s))\, ds \\ &\leqslant -\lambda(t_{k+1} - t_k) = -\lambda\frac{2L}{\xi}. \end{aligned} \tag{4.2.4}$$

On the other hand, if it happens that, for $t_k \leqslant t_1 \leqslant t_{k+1}$,

$$\delta < \| x(t_1)\| < \rho, \qquad d[x(t_1), E] < \tfrac{1}{2}\eta,$$

then there exists a t_3, $t_1 \leqslant t_3 \leqslant t_1 + 2L/\xi$ such that $d[x(t_3), E] = \eta$, in view of (2). It follows that there also exists a t_4, $t_1 \leqslant t_4 < t_3$ satisfying $d[x(t_4), E] = \frac{1}{2}\eta$. These considerations lead to $d[x(t_3), x(t_4)] \geqslant \frac{1}{2}\eta$, and hence we obtain, because of (1),

$$\tfrac{1}{2}\eta \leqslant Mn^{1/2}(t_3 - t_4),$$

which implies

$$\frac{\eta}{2Mn^{1/2}} \leqslant t_3 - t_4 \leqslant \frac{2L}{\xi}. \tag{4.2.5}$$

Moreover,

$$n(t_3) - n(t_1) \leqslant \int_{t_1}^{t_4} D^+V_1(s, x(s))\, ds + \int_{t_4}^{t_3} D^+V_1(s, x(s))\, ds$$
$$\leqslant -\lambda(t_3 - t_4) \leqslant \frac{-\lambda\eta}{2Mn^{1/2}} = -\lambda_1 .$$

Since $n(t)$ is a nonincreasing function, we have

$$n(t_{k+2}) \leqslant n(t_3) \leqslant n(t_1) - \lambda_1$$
$$\leqslant n(t_k) - \lambda_1 .$$

Also, on the basis of (4.2.5), we obtain from (4.2.4) that

$$n(t_{k+2}) \leqslant n(t_k) - \lambda_1 .$$

Thus, in any case,

$$V_1(t_{k+2}, x(t_{k+2})) \leqslant V_1(t_k, x(t_k)) - \lambda_1 .$$

Choose an integer k^* such that $\lambda_1 k^* > a(\delta_0)$ and $T = T(\epsilon) = 4k^*L/\xi(\epsilon)$. Assume that, for $t_0 \leqslant t \leqslant t_0 + T$,

$$\| x(t, t_0, x_0)\| \geqslant \delta.$$

It then results from the preceding considerations that

$$V_1(t_0 + T, x(t_0 + T)) \leqslant V_1(t_0, x_0) - k^*\lambda_1$$
$$\leqslant a(\delta_0) - k^*\lambda_1$$
$$\leqslant 0,$$

which is incompatible with the positive definiteness of $V_1(t, x)$. Thus, there exists a $t^* \in [t_0, t_0 + T]$ satisfying

$$\| x(t^*, t_0, x_0)\| < \delta,$$

and the proof is complete.

4.3. Instability

In Sect. 3.3, we proved a theorem on instability by means of a single Lyapunov function. We give below an instability theorem in which two Lyapunov functions are used.

THEOREM 4.3.1. Suppose that the following conditions hold:

(i) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $f(t, x)$ is bounded on $J \times S_\rho$.

(ii) $V_1 \in C[J \times S_\rho, R_+]$, $V_1(t, x)$ is locally Lipschitzian in x, decrescent, and, for any $t \geqslant 0$, it is possible to find points x lying in any given small neighborhood of the origin such that $V_1(t, x) > 0$.

(iii) $D^+V_1(t, x) \geqslant 0$, $(t, x) \in J \times S_\rho$, and, in each domain $t \geqslant 0$, $\alpha < \| x \| < \rho$, $D^+V_1(t, x) \geqslant \phi_\alpha(t)\, w(x)$, where $w(x) \geqslant 0$ is continuous for $x \in S_\rho$ and $\phi_\alpha(t) \geqslant 0$ is continuous in t such that, for any infinite system S of closed, nonintersecting intervals of J of an identical fixed interval, we have

$$\int_S \phi_\alpha(s)\, ds = \infty. \tag{4.3.1}$$

(iv) $V_2 \in C[J \times S_\rho, R_+]$, and $V_2(t, x)$ is bounded on $J \times S_\rho$ and is locally Lipschitzian in x. Furthermore, given any number α, $0 < \alpha < \rho$, it is possible to find $\eta = \eta(\alpha)$, $\eta < \alpha$, and a continuous function $\xi_\alpha(t) > 0$ such that

$$\int_t^\infty \xi_\alpha(t)\, dt = \infty, \tag{4.3.2}$$

and, in the set $\alpha < \| x \| < \rho$, $d(x, E) < \eta$, $t \in J$,

$$D^+V_2(t, x) \geqslant \xi_\alpha(t), \tag{4.3.3}$$

where

$$E = [x \in S_\rho : w(x) = 0].$$

Then, the trivial solution of (4.1.1) is unstable.

Proof. The proof of this theorem closely resembles that of Theorem 4.2.1, and hence we shall be brief. Suppose that, under the conditions of the theorem, the trivial solution is stable. That is, given $0 < \epsilon < \rho$, $t_0 \in J$, there exists a $\delta > 0$ such that $\| x_0 \| < \delta$ implies $\| x(t, t_0, x_0) \| < \epsilon$, $t \geqslant t_0$.

According to assumption (ii), a point (t_0, x_0^*) can be found such that $\| x_0^* \| < \delta$ and $V_1(t_0, x_0^*) > 0$. We shall consider the motion $x(t) = x(t, t_0, x_0^*)$ and its properties:

(1) $d(x(t), x(\tau)) \geqslant \eta$, $t > \tau$; then $t - \tau \geqslant \eta / M n^{1/2}$. This is clear from (1) in the proof of Theorem 4.2.1.

(2) For every $t \geqslant t_0$, there will be a positive number α such that

$$\alpha < \| x(t) \| < \epsilon < \rho. \tag{4.3.4}$$

This is compatible with the assumption of stability, that is, $\| x(t)\| < \epsilon$, $t \geqslant t_0$. However, since $D^+V_1(t, x) \geqslant 0$, it follows that

$$V_1(t, x(t)) \geqslant V_1(t_0, x_0^*) > 0.$$

Since $V_1(t, x)$ is decrescent, for numbers $V_1(t_0, x_0^*) > 0$, a number $\alpha > 0$ can be found such that, for all $t \geqslant t_0$, $\| x \| \leqslant \alpha$, we shall have

$$V_1(t, x) < V_1(t_0, x_0^*).$$

Consequently, $\| x \| < \alpha$ is not possible. According to (iv), there exists a number $\eta = \eta(\alpha, \epsilon)$, $\eta < \alpha$, and a continuous function $\xi_\alpha(t) > 0$ such that (4.3.2) and (4.3.3) hold.

(3) If $d(x(\tau), E) < \eta$, then a $t^* > \tau$ can be found such that

$$d(x(t^*), E) = \eta. \tag{4.3.5}$$

Suppose that $d(x(t), E) < \eta$ for all $t \geqslant \tau$. Letting $m(t) = V_2(t, x(t))$, we obtain, using the Lipschitzian character of $V_2(t, x)$ in x, the inequality

$$D^+m(t) \geqslant D^+V_2(t, x(t)) \geqslant \xi_\alpha(t),$$

and hence

$$m(t) + m(\tau) \geqslant \int_\tau^t D^+m(s)\, ds \geqslant \int_\tau^t \xi_\alpha(s)\, ds.$$

Since $V_2(t, x)$ is assumed to be bounded, the relation (4.3.2) shows that $d(x(t), E) < \eta$ cannot hold for all $t \geqslant \tau$. Hence, there exists a $t^* > \tau$ such that (4.3.5) is satisfied.

(4) *If* $d(x(\tau), E) < \eta/2$, then, for $t = t^*$, when $d(x(t^*), E) = \eta$, we have

$$V_1(t^*, x(t^*)) \geqslant V_1(\tau, x(\tau)) + \epsilon \int_{t^{**}}^{t^*} \phi_\alpha(s)\, ds,$$

where

$$\tau \leqslant t^{**} = t^* - \frac{\eta}{2Mn^{1/2}}$$

and

$$\epsilon = \inf[w(x), \alpha < \| x \| < \rho, d(x, E) \geqslant \tfrac{1}{2}\eta] > 0.$$

In fact, under the given conditions, $\tau < t_* < t^*$ can be found such that

$$d(x(t_*), E) = \tfrac{1}{2}\eta,$$

and, for $t_* \leqslant t \leqslant t^*$, we shall have

$$\tfrac{1}{2}\eta \leqslant d(x(t), E) \leqslant \eta.$$

Hence, by (iii), it follows that

$$D^+V_1(t, x(t)) \geqslant \phi_\alpha(t)\, w(x(t)) \geqslant \epsilon\phi_\alpha(t),$$

using the fact that $V_1(t, x)$ is locally Lipschitzian in x, and, consequently,

$$V_1(t^*, x(t^*)) \geqslant V_1(\tau, x(\tau)) + \epsilon \int_{t_*}^{t^*} \phi_\alpha(s)\, ds.$$

Observing, however, that $d(x(t^*), x(t_*)) \geqslant \frac{1}{2}\eta$, we get, in view of (1), that

$$t^* - \tau \geqslant t^* - t_* \geqslant \frac{\eta}{2Mn^{1/2}}.$$

(5) There is no number $t_1 \geqslant t_0$ such that, for all $t > t_1$, we would have

$$d(x(t), E) \geqslant \tfrac{1}{2}\eta.$$

Indeed, if such a t_1 exists, then, for all $t > t_1$, we should have

$$\begin{aligned} V_1(t, x(t)) &= V_1(t_1, x(t_1)) + \int_{t_1}^{t} D^+V_1(s, x(s))\, ds \\ &\geqslant V_1(t_1, x(t_1)) + \epsilon \int_{t_1}^{t} \phi_\alpha(s)\, ds. \end{aligned}$$

By (4.3.1), this implies that $V_1(t, x(t)) \to \infty$ as $t \to \infty$, which is absurd because of the relation (4.3.4) and the fact that $V_1(t, x)$ is decrescent. Thus it follows that, for any t_i^*, a $\tau_{i+1} > t_i^*$ can be found such that

$$d(x(\tau_{i+1}), E) < \tfrac{1}{2}\eta,$$

and, according to (3), there corresponds a $t_{i+1}^* > \tau_{i+1}$ satisfying

$$d(x(t_{i+1}^*), E) = \eta.$$

Let us consider the infinite sequence of numbers

$$t_0 < \tau_1 < t_1^* < \cdots < \tau_i < t_i^* < \cdots.$$

In view of assumption (iii) and (4), we have

$$V_1(t_i^*, x(t_i^*)) \geqslant V_1(t_0, x_0) + \epsilon \sum_{j=1}^{i} \int_{t_j^{**}}^{t_j^*} \phi_\alpha(s)\, ds,$$

where $\tau_j \leqslant t_j^{**} = t_j^* - \eta/2Mn^{1/2}$. The infinite system of segments $[t_j^{**}, t_j^*]$ satisfies condition (iii), and therefore the last sum increases

indefinitely with i. In other words, $V_1(t_i^*, x(t_i^*)) \to \infty$ as $i \to \infty$. This is not compatible with the boundedness of $V_1(t, x(t))$. The contradiction shows that the assumption of stability is wrong, and the theorem is proved.

4.4. Conditional stability and boundedness

Let, for $k < n$, $M_{(n-k)}$ denote a manifold of $(n - k)$ dimensions containing the origin. Let $S(\alpha)$, $\bar{S}(\alpha)$ represent the sets

$$S(\alpha) = [x \in S_\rho : \| x \| < \alpha],$$

$$\bar{S}(\alpha) = [x \in S_\rho : \| x \| \leqslant \alpha],$$

respectively. Suppose that $x(t) = x(t, t_0, x_0)$ is any solution of (4.1.1). Then, corresponding to the stability and boundedness definitions (S_1)–(S_8) and (B_1)–(B_8), we shall designate the concepts of conditional stability and boundedness by (C_1)–(C_{16}). We shall define (C_1) only, since, on that basis, other definitions may be formulated.

DEFINITION 4.4.1. The trivial solution of (4.1.1) is said to be (C_1) *conditionally equistable* if, for each $\epsilon > 0$ and $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that

$$x(t, t_0, x_0) \subset S(\epsilon), \qquad t \geqslant t_0,$$

provided

$$x_0 \in \bar{S}(\delta) \cap M_{(n-k)}.$$

Evidently, if $k = 0$ so that $M_{(n-k)} = R^n$, definitions (C_1)–(C_{16}) coincide with the stability and boundedness notions (S_1)–(S_8) and (B_1)–(B_8).

Analogous to the definitions (C_1)–(C_{16}), we need some kind of of conditional stability and boundedness concepts with respect to the auxiliary differential system (4.1.4). Perhaps the simplest type of definition is the following.

DEFINITION 4.4.2. The trivial solution of the system (4.1.4) is said to be (C_1^*) *conditionally equistable* if, for each $\epsilon > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that the condition

$$\sum_{i=1}^{N} u_{i0} \leqslant \delta, \qquad \text{and} \quad u_{i0} = 0 \qquad (i = 1, 2, \ldots, k)$$

implies

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < \epsilon, \qquad t \geqslant t_0 .$$

Definitions (C_2^*)–(C_{16}^*) are to be understood in a similar way.

THEOREM 4.4.1. Assume that

(i) $g \in C[J \times R_+^N, R^N]$, $g(t, 0) \equiv 0$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(ii) $V \in C[J \times S_\rho, R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, $\sum_{i=1}^N V_i(t, x)$ is positive definite, and

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \qquad \text{as} \quad \| x \| \to 0 \quad \text{for each} \quad t \in J;$$

(iii) $V_i(t, x) \equiv 0$ $(i = 1, 2,..., k)$, $k < n$, if $x \in M_{(n-k)}$, where $M_{(n-k)}$ is an $(n - k)$ dimensional manifold containing the origin;

(iv) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times S_\rho .$$

Then, if the trivial solution of (4.1.4) is conditionally equistable, the trivial solution of the system (4.1.1) is conditionally equistable.

Proof. Let $0 < \epsilon < \rho$ and $t_0 \in J$ be given. Since $\sum_{i=1}^N V_i(t, x)$ is positive definite, there exists a $b \in \mathscr{K}$ such that

$$b(\| x \|) \leqslant \sum_{i=1}^{N} V_i(t, x), \qquad (t, x) \in J \times S_\rho . \tag{4.4.1}$$

Assume that the trivial solution of the auxiliary system (4.1.4) is conditionally equistable. Then, given $b(\epsilon) > 0$ and $t_0 \in J$, there exists a $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, so that

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 , \tag{4.4.2}$$

provided

$$\sum_{i=1}^{N} u_{i0} \leqslant \delta, \qquad u_{i0} = 0 \qquad (i = 1, 2,..., k). \tag{4.4.3}$$

Let us choose $u_{i0} = V_i(t_0, x_0)$ $(i = 1, 2,..., N)$ and $x_0 \in M_{(n-k)}$ so that $u_{i0} = 0$ $(i = 1, 2,..., k)$, by condition (iii). Furthermore, since $\sum_{i=1}^N V_i(t, x) \to 0$ as $\| x \| \to 0$ for each $t \in J$, and $V(t, x)$ is continuous,

it is possible to find a $\delta_1 = \delta_1(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, verifying the inequalities

$$\| x_0 \| \leqslant \delta_1, \qquad \sum_{i=1}^{N} V_i(t_0, x_0) \leqslant \delta \tag{4.4.4}$$

simultaneously. With this choice, it certainly follows that

$$x_0 \in \bar{S}(\delta_1) \cap M_{(n-k)}$$

implies $x(t, t_0, x_0) \subset S(\epsilon)$, $t \geqslant t_0$. If this were not true, there would exist a $t_1 > t_0$ and a solution $x(t, t_0, x_0)$ of (4.1.1) such that, whenever $x_0 \in \bar{S}(\delta_1) \cap M_{(n-k)}$, we have $x(t, t_0, x_0) \subset S(\epsilon)$, $t \in [t_0, t_1)$, and $x(t_1, t_0, x_0)$ lies on the boundary of $S(\epsilon)$. This means that

$$\| x(t_1, t_0, x_0)\| = \epsilon, \qquad \| x(t, t_0, x_0)\| < \rho, \qquad t \in [t_0, t_1],$$

and, consequently,

$$b(\epsilon) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1, t_0, x_0)). \tag{4.4.5}$$

Moreover, for $t \in [t_0, t_1]$, we can apply Theorem 4.1.1 to obtain

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0), \qquad t \in [t_0, t_1],$$

where $r(t, t_0, u_0)$ is the maximal solution of (4.1.4), which implies that

$$\sum_{i=1}^{N} V_i(t, x(t, t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0), \qquad t \in [t_0, t_1]. \tag{4.4.6}$$

Notice that, from the choice $u_{i0} = V_i(t_0, x_0)$ and the relation (4.4.4), $x_0 \in \bar{S}(\delta_1) \cap M_{(n-k)}$ assures that (4.4.3) is satisfied. Hence, (4.4.2) and (4.4.6) yield the inequality

$$\sum_{i=1}^{N} V_i(t_1, x(t_1, t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0, u_0) < b(\epsilon),$$

which is incompatible with (4.4.5). Thus, $x(t, t_0, x_0) \subset S(\epsilon)$, $t \geqslant t_0$, provided $x_0 \in \bar{S}(\delta_1) \cap M_{(n-k)}$, and the theorem is proved.

THEOREM 4.4.2. Let assumptions (i), (ii), (iii), and (iv) of Theorem 4.4.1 hold. Suppose further that

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \qquad \text{as} \quad \| x \| \to 0 \quad \text{uniformly in} \quad t. \tag{4.4.7}$$

Then the conditional uniform stability of the solution $u = 0$ of (4.1.4) guarantees the conditional uniform stability of the trivial solution of (4.1.1).

Proof. By definition (C_2^*), it is evident that δ occurring in (4.4.3) is independent of t_0. In view of (4.4.7), this makes it possible to choose δ_1 also independent of t_0, according to (4.4.4). Noting these changes, the theorem can be proved as in Theorem 4.4.1.

THEOREM 4.4.3. Under assumptions (i), (ii), (iii), and (iv) of Theorem 4.4.1, the conditional equi-asymptotic stability of the trivial solution of (4.1.4) implies the conditional equi-asymptotic stability of the trivial solution of the system (4.1.1).

Proof. Assume that the trivial solution of the auxiliary system (4.1.4) is conditionally equi-asymptotically stable. Then, it is conditionally equistable and conditionally quasi-equi-asymptotically stable. Since, by Theorem 4.4.1, the conditional equistability of the trivial solution of (4.1.1) is guaranteed, we need only to prove the conditional quasi-equi-asymptotic stability of the solution $x = 0$ of (4.1.1). For this purpose, suppose that we are given $0 < \epsilon < \rho$ and $t_0 \in J$. Then, given $b(\epsilon) > 0$ and $t_0 \in J$, there exist two positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that, if the condition

$$\sum_{i=1}^{N} u_{i0} \leqslant \delta_0, \qquad u_{i0} = 0 \qquad (i = 1, 2,..., k) \tag{4.4.8}$$

holds, we have

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 + T. \tag{4.4.9}$$

As previously, the choice $u_{i0} = V_i(t_0, x_0)$ and $x_0 \in M_{(n-k)}$ implies $u_{i0} = 0$ $(i = 1, 2,..., k)$. Also, there exists a $\tilde{\delta}_0 = \tilde{\delta}_0(t_0)$ satisfying

$$\| x_0 \| \leqslant \tilde{\delta}_0, \qquad \sum_{i=1}^{N} V_i(t_0, x_0) \leqslant \delta_0 \tag{4.4.10}$$

at the same time. Let $\bar{\delta}_0 = \min[\tilde{\delta}_0, \delta_0^*]$, where $\delta_0^* = \delta(t_0, \rho)$ is the number obtained by taking $\epsilon = \rho$. Thus, if $x_0 \in \bar{S}(\bar{\delta}_0) \cap M_{(n-k)}$, we notice that the condition (4.4.8) is fulfilled. Furthermore, since (C_1) holds, the inequality (4.4.6) is valid for all $t \geqslant t_0$. We can now assert that $x(t) \subset S(\epsilon)$, $t \geqslant t_0 + T$ whenever $x_0 \in \bar{S}(\bar{\delta}_0) \cap M_{(n-k)}$. For otherwise, suppose that there exists a sequence $\{t_k\}$, $t_k \geqslant t_0 + T$, and $t_k \to \infty$

as $k \to \infty$ such that, for some solution $x(t, t_0, x_0)$ of (4.1.1) with $x_0 \in \bar{S}(\delta_0) \cap M_{(n-k)}$, we have

$$\| x(t_k, t_0, x_0)\| \geqslant \epsilon.$$

This leads to an absurdity,

$$b(\epsilon) \leqslant \sum_{i=1}^{N} V_i(t_k, x(t_k, t_0, x_0))$$

$$\leqslant \sum_{i=1}^{N} r_i(t_k, t_0, u_0) < b(\epsilon),$$

in view of relations (4.4.1), (4.4.6), and (4.4.9). We thus have (C_3), and consequently the theorem is established.

THEOREM 4.4.4. Suppose that assumptions (i), (ii), (iii), and (iv) of Theorem 4.4.1 hold, together with (4.4.7). Then, the conditional uniform asymptotic stability of the trivial solution of (4.1.4) implies that the trivial solution of (4.1.1) is conditionally uniformly asymptotically stable.

Proof. We proceed as in Theorem 4.4.3, observing that the uniformity of conditional stability is assured by Theorem 4.4.2. As the numbers δ_0 and T are independent of t_0, δ_0 resulting from (4.4.10) is certainly independent of t_0, because of (4.4.7).

THEOREM 4.4.5. Assume that

(i) $g \in C[J \times R_+^N, R^N]$ and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(ii) $V \in C[J \times R^n, R^N]$, $V(t, x)$ is locally Lipschitzian in x, and

$$b(\| x \|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times R^n,$$

where $a, b \in \mathscr{K}$ on the interval $0 \leqslant u < \infty$, and

$$b(u) \to \infty \qquad \text{as} \quad u \to \infty;$$

(iii) $V_i(t, x) \equiv 0$, $i = 1, 2,..., k$, $k < n$, if $x \in M_{(n-k)}$, where $M_{(n-k)}$ is an $(n - k)$ dimensional manifold containing the origin;

(iv) $f \in C[J \times R^n, R^n]$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times R^n.$$

Then, if the auxiliary system (4.1.4) satisfies one of the definitions (C_9^*)–(C_{16}^*), the system (4.1.1) verifies the corresponding one of the definitions (C_9)–(C_{16}). On the basis of parallel theorems of Sect. 3.13 and the proofs given previously, the proof of the respective statements of this theorem can be constructed.

Let us now indicate the modifications necessary in order to obtain the usual stability and boundedness results, using several Lyapunov functions. Designate by (C_1^{**})–(C_{16}^{**}) the parallel definitions obtained by dropping the conditional character in (C_1^*)–(C_{16}^*). For example, the definitions (C_1^{**}) would run as follows:

(C_1^{**}) For each $\epsilon > 0$ and $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that the inequality

$$\sum_{i=1}^{N} u_{i0} \leqslant \delta$$

implies

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < \epsilon, \qquad t \geqslant t_0 .$$

As a typical example, we shall merely state a theorem that gives sufficient conditions, in terms of any Lyapunov function, for the equistability of the trivial solution of (4.1.1).

THEOREM 4.4.6. Suppose that

(i) $g \in C[J \times R_+^N, R^N]$, $g(t, 0) \equiv 0$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(ii) $V \in C[J \times S_\rho, R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, $\sum_{i=1}^N V_i(t, x)$ is positive definite, and

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \qquad \text{as} \quad \| x \| \to 0 \quad \text{for each} \quad t \in J;$$

(iii) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times S_\rho .$$

Then the definition (C_1^{**}) implies that the trivial solution of (4.1.1) is equistable.

To exhibit the fruitfulness of using vector Lyapunov function, even in the case of ordinary stability, we give the following example.

Example. Let us consider the two systems

$$\begin{aligned} x' &= e^{-t}x + y\sin t - (x^3 + xy^2)\sin^2 t, \\ y' &= x\sin t + e^{-t}y - (x^2y + y^3)\sin^2 t. \end{aligned} \tag{4.4.11}$$

Suppose we choose a single Lyapunov function V given by

$$V(t, x) = x^2 + y^2.$$

Then, it is evident that

$$D^+V(t, x) \leqslant 2(e^{-t} + |\sin t|)\, V(t, x),$$

using the inequality $2|ab| \leqslant a^2 + b^2$ and observing that

$$[x^2 + y^2]^2 \sin^2 t \geqslant 0.$$

Clearly, the trivial solution of the scalar differential equation

$$u' = 2(e^{-t} + |\sin t|)\, u, \qquad u(t_0) = u_0 \geqslant 0$$

is not stable, and so we cannot deduce any information about the stability of the trivial solution of (4.4.11) from Theorem 3.3.1, although it is easy to check that it is stable. On the other hand, let us attempt to seek a Lyapunov function as a quadratic form with constant coefficients

$$V(t, x) = \tfrac{1}{2}[x^2 + 2Bxy + Ay^2]. \tag{4.4.12}$$

Then, the function $D^+V(t, x)$ with respect to (4.4.11) is equal to the sum of two functions $w_1(t, x)$, $w_2(t, x)$, where

$$\begin{aligned} w_1(t, x) &= x^2[e^{-t} + B\sin t] + xy[2Be^{-t} + (A + 1)\sin t] \\ &\quad + y^2[Ae^{-t} + B\sin t], \\ w_2(t, x) &= -\sin^2 t[(x^2 + y^2)(x^2 + 2Bxy + Ay^2)]. \end{aligned}$$

For arbitrary A and B, the functions $V(t, x)$ defined in (4.4.12) does not satisfy Lyapunov's theorem (Corollary 3.3.2) on the stability of motion. Let us try to satisfy the conditions of Theorem 3.3.3 by assuming $w_1(t, x) = \lambda(t)\, V(t, x)$. This equality can occur in two cases:

(i) $A_1 = 1$, $B_1 = 1$, $\lambda_1(t) = 2[e^{-t} + \sin t]$ when $V_1(t, x) = \frac{1}{2}(x + y)^2$.

(ii) $A_2 = 1$, $B_2 = -1$, $\lambda_2(t) = 2[e^{-t} - \sin t]$ when $V_2(t, x) = \frac{1}{2}(x - y)^2$.

The functions V_1, V_2 are not positive definite and hence do not satisfy Theorem 3.3.3. However, they do fulfill the conditions of Theorem 4.4.6. In fact,

(a) the functions $V_1(t, x) \geqslant 0$, $V_2(t, x) \geqslant 0$, and $\sum_{i=1}^{2} V_i(t, x) = x^2 + y^2$, and therefore $\sum_{i=1}^{2} V_i(t, x)$ is positive definite as well as decrescent;

(b) the vectorial inequality $D^+V(t, x) \leqslant g(t, V(t, x))$ is satisfied with the functions

$$g_1(t, u_1, u_2) = 2(e^{-t} + \sin t)\, u_1,$$

$$g_2(t, u_1, u_2) = 2(e^{-t} - \sin t)\, u_2.$$

It is clear that $g(t, u)$ is quasi-monotone nondecreasing in u, and the null solution of $u' = g(t, u)$ is stable. Consequently, the trivial solution of (4.4.11) is stable by Theorem 4.4.6.

4.5. Converse theorems

We shall consider the converse problem of showing the existence of several Lyapunov functions, whenever the motion is conditionally stable or asymptotically stable. The techniques employed in the construction of a single Lyapunov function earlier in Sect. 3.6 do not right away extend to this situation. As will be seen, the results rest heavily on the choice of special solutions of a certain differential system and the chain of inequalities among them, a kind of diagonal selection of the components of these solutions, and the quasi-monotone property.

With a view to avoid interruption in the proofs, let us first exhibit some properties of certain solutions of the system (4.1.4) and its related system

$$u' = g^*(t, u), \tag{4.5.1}$$

where

$$g^*(t, u) = \begin{pmatrix} g_1(t, u_1, u_2, \dots, u_N) \\ g_2(t, 0, u_2, \dots, u_N) \\ \cdots \\ g_i(t, 0, 0, \dots, u_i, \dots, u_N) \\ \cdots \\ g_N(t, 0, 0, \dots, 0, u_N) \end{pmatrix}.$$

Assume that $g \in C[J \times R_+^N, R^N]$, $g(t, 0) \equiv 0$, $\partial g(t, u)/\partial u$ exists and is continuous for $(t, u) \in J \times R_+^N$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$. Evidently, $g^*(t, u)$ also satisfies these

assumptions. Moreover, since $u_i \geqslant 0$ $(i = 1, 2, \ldots, N)$, it follows, in view of the quasi-monotone property of $g(t, u)$, that

$$g^*(t, u) \leqslant g(t, u). \tag{4.5.2}$$

Observe that the hypothesis on $g(t, u)$ guarantees the existence and uniqueness of solutions of (4.1.4) as well as their continuous dependence on initial values. Also, the solutions $u(t, t_0, u_0)$ are continuously differentiable with respect to the initial values. Furthermore, $u \equiv 0$ is the trivial solution of (4.1.4). Clearly, similar assertions can be made with respect to the related system (4.5.1).

If $U(t) = U(t, 0, u_0)$ and $U^*(t) = U^*(t, 0, u_0)$ are the solutions of (4.1.4) and (4.5.1), through the same point $(0, u_0)$, respectively, it follows, from Corollary 1.7.1, that

$$U^*(t) \leqslant U(t), \qquad t \geqslant 0, \tag{4.5.3}$$

in view of (4.5.2).

Consider next the N initial vectors, with $u_{i0} \geqslant 0$ $(i = 1, 2, \ldots, N)$ defined by

$$\begin{aligned} p_1 &= (u_{10}, 0, \ldots, 0), \\ p_2 &= (u_{10}, u_{20}, 0, \ldots, 0), \\ &\cdots \\ p_i &= (u_{10}, u_{20}, \ldots, u_{i0}, 0, \ldots, 0), \\ &\cdots \\ p_N &= (u_{10}, u_{20}, \ldots, u_{N0}). \end{aligned}$$

It is easy to see that $p_i \leqslant p_{i+1}$, for each $i = 1, 2, \ldots, N - 1$. Let us denote the solution of the system (4.5.1) through the point $(0, p_i)$ by

$$U_i^*(t) = U_i^*(t, 0, p_i) = \begin{pmatrix} u_{i1}(t, 0, p_i) \\ u_{i2}(t, 0, p_i) \\ \cdots \\ \cdots \\ u_{iN}(t, 0, p_i) \end{pmatrix}$$

for each fixed i, $i = 1, 2, \ldots, N$.

By Corollary 1.7.1, it follows that

$$U_i^*(t) \leqslant U_{i+1}^*(t), \qquad t \geqslant 0,$$

where $U_i^*(t)$, $U_{i+1}^*(t)$ are the solutions of the system (4.5.1) through $(0, p_i)$ and $(0, p_{i+1})$, respectively.

This implies that, for each $j = 1, 2, \ldots, N$ and $t \geqslant 0$,

$$u_{1j}(t, 0, p_1) \leqslant u_{2j}(t, 0, p_2) \leqslant \cdots \leqslant u_{Nj}(t, 0, p_N). \tag{4.5.4}$$

We may now have the following:

THEOREM 4.5.1. Assume that

(i) the function $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and $\partial f(t, x)/\partial x$ exists and is continuous for $(t, x) \in J \times S_\rho$;

(ii) the solution $x(t, 0, x_0)$ of the system (4.1.1) satisfies the estimate

$$\begin{aligned} \beta_1(\| x_0 \|) &\leqslant \| x(t, 0, x_0)\| \\ &\leqslant \beta_2(\| x_0 \|), \qquad t \geqslant 0, \qquad x_0 \in M_{(n-k)}, \end{aligned} \tag{4.5.5}$$

where $\beta_1, \beta_2 \in \mathscr{K}$;

(iii) the function $g \in C[J \times R_+^N, R^N]$, $g(t, 0) \equiv 0$, $\partial g(t, u)/\partial u$ exists and is continuous for $(t, u) \in J \times R_+^N$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(iv) the solution $U(t, 0, p_N)$ of the system (4.1.4) verifies the inequality

$$\sum_{i=1}^{N} u_i(t, 0, p_N) \leqslant \gamma_2 \left(\sum_{i=1}^{N} u_{i0} \right), \qquad t \geqslant 0, \tag{4.5.6}$$

provided $u_{i0} = 0$ $(i = 1, 2, \ldots, k)$, where $\gamma_2 \in \mathscr{K}$;

(v) the solution $U_N^*(t, 0, p_N)$ of the related system (4.5.1) is such that

$$u_{NN}(t, 0, p_N) \geqslant \gamma_1 \left(\sum_{i=1}^{N} u_{i0} \right), \qquad t \geqslant 0, \tag{4.5.7}$$

if $u_{i0} = 0$ $(i = 1, 2, \ldots, k)$, where $\gamma_1 \in \mathscr{K}$. Then, there exists a vector function $V(t, x)$ with the following properties:

(1) $V \in C[J \times S_\rho, R_+^N]$, and $V(t, x)$ possesses continuous partial derivatives with respect to t and the components of x for $(t, x) \in J \times S_\rho$;

(2) $$\begin{aligned} V_i'(t, x) &= \frac{\partial V_i(t, x)}{\partial t} + \frac{\partial V_i(t, x)}{\partial x} \cdot f(t, x) \\ &\leqslant g_i(t, V(t, x)) \qquad (i = 1, 2, \ldots, N); \end{aligned}$$

(3) $$V_i(t, x) = 0 \quad \text{if} \quad x \in M_{(n-k)}, \qquad i = 1, 2, \ldots, k;$$

(4) $$b(\| x \|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant a(\| x \|), \qquad (t, x) \in J \times S \,,$$

where $a, b \in \mathscr{K}$.

Proof. Let us first observe that assumption (i) implies the existence and uniqueness of solutions of (4.1.1), as well as their continuous dependence on the initial values. Also, the solutions $x(t, t_0, x_0)$ are continuously differentiable functions with respect to the initial values (t_0, x_0), and the system (4.1.1) possesses the trivial solution.

Let us denote $x(t, 0, x_0)$ by x so that, by uniqueness of solutions, we have $x_0 = x(0, t, x)$. Choose any continuous function $\mu(x) \in R_+^N$, possessing continuous partial derivatives $\partial\mu(x)/\partial x$ for $x \in S_\rho$, such that

$$\alpha_1(\| x \|) \leqslant \sum_{i=1}^{N} \mu_i(x) \leqslant \alpha_2(\| x \|), \tag{4.5.8}$$

where $\alpha_1, \alpha_2 \in \mathscr{K}$ and

$$\mu_i(x) = 0 \qquad (i = 1, 2,..., k) \qquad \text{if} \quad x \in M_{(n-k)}. \tag{4.5.9}$$

We then define the vector function $V(t, x)$ as follows:

$$\begin{aligned} V_1(t, x) &= u_{11}(t, 0, \mu_1(x(0, t, x)), 0,..., 0),\\ V_2(t, x) &= u_{22}(t, 0, \mu_1(x(0, t, x)), \mu_2(x(0, t, x)), 0,..., 0),\\ &\cdots\\ V_N(t, x) &= u_{NN}(t, 0, \mu_1(x(0, t, x)),..., \mu_N(x(0, t, x))). \end{aligned} \tag{4.5.10}$$

Because of the continuity of the functions $x(0, t, x)$, $\mu(x)$, $U_1^*(t)$,..., $U_N^*(t)$, with respect to their arguments, it is clear that $V_i(t, x)$ $(i = 1, 2,..., N)$ is defined and continuous for $(t, x) \in J \times S_\rho$. Since the functions f and g (and hence g^*) satisfy hypotheses (i) and (iii), the functions $U_1^*(t)$, $U_2^*(t)$,..., $U_N^*(t)$, and $x(0, t, x)$ are all continuously differentiable with respect to their arguments. This, together with the choice of $\mu(x)$, shows that $V(t, x)$ possesses continuous partial derivatives with respect to t and the component of x. Thus, for each $i = 1, 2,..., N$,

$$\begin{aligned} V_i'(t, x) &= u_{ii}'(t, 0, \mu_1(x(0, t, x)),..., \mu_i(x(0, t, x)), 0,..., 0)\\ &\quad + \frac{\partial u_{ii}}{\partial u_0} \cdot \frac{\partial \mu}{\partial x} \cdot \left[\frac{\partial x(0, t, x)}{\partial t_0} + \frac{\partial x(0, t, x)}{\partial x_0} \cdot f(t, x)\right]\\ &= g_i(t, 0,..., 0, u_{ii}[t, 0, \mu_1(x(0, t, x)),..., \mu_i(x(0, t, x)),\\ &\quad 0,..., 0],..., u_{iN}[t, 0, \mu_1(x(0, t, x)),..., u_i(x(0, t, x)), 0,..., 0]), \end{aligned}$$

since, by relation (3.6.11),

$$\frac{\partial x(0, t, x)}{\partial t_0} + \frac{\partial x(0, t, x)}{\partial x_0} \cdot f(t, x) = 0.$$

Using the quasi-monotone nondecreasing character of $g(t, u)$ in u, the fact that the solutions $U_1^*(t),\dots, U_N^*(t)$ are all nonnegative, and relations (4.5.4), we obtain

$$\begin{aligned} V_i'(t, x) &\leqslant g_i[t, u_{11}(t, 0, \mu_1(x(0, t, x)), 0,\dots, 0),\dots,\\ &\qquad u_{ii}(t, 0, \mu_1(x(0, t, x)),\dots, \mu_i(x(0, t, x)), 0,\dots, 0),\dots,\\ &\qquad u_{NN}(t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))]\\ &= g_i[t, V_1(t, x), V_2(t, x),\dots, V_N(t, x)]\\ &= g_i(t, V(t, x)). \end{aligned}$$

This proves (1) and (2).

To show that (3) holds, we observe that, if x belongs to $M_{(n-k)}$, then $x_0 = x(0, t, x)$ also belongs to the manifold $M_{(n-k)}$. Now, by the definition (4.5.10), the choice of $\mu(x)$ satisfying (4.5.9), and the fact that the system (4.5.1) has the identically zero solution, it follows that, if $x \in M_{(n-k)}$, $V_i(t, x) \equiv 0$ $(i = 1, 2,\dots, k)$.

Since $x = x(t, 0, x_0)$ and $x_0 = x(0, t, x)$, we get, from (4.5.5), that

$$\beta_2^{-1}(\| x \|) \leqslant \| x(0, t, x)\| \leqslant \beta_1^{-1}(\| x \|), \tag{4.5.11}$$

where $\beta_2^{-1}, \beta_1^{-1} \in \mathscr{K}$. The definition (4.5.10) and the relations (4.5.4) yield

$$\begin{aligned} \sum_{i=1}^{N} V_i(t, x) &= u_{11}[t, 0, \mu_1(x(0, t, x)), 0,\dots, 0]\\ &\qquad + u_{22}[t, 0, \mu_1(x(0, t, x)), \mu_2(x(0, t, x)), 0,\dots, 0]\\ &\qquad + \cdots + \cdots\\ &\qquad + u_{NN}[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))]\\ &\leqslant u_{N1}[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))]\\ &\qquad + u_{N2}[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))]\\ &\qquad + \cdots + \cdots\\ &\qquad + u_{NN}[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))], \end{aligned}$$

which, by virtue of (4.5.3), leads to

$$\begin{aligned} \sum_{i=1}^{N} V_i(t, x) &\leqslant u_1[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))]\\ &\qquad + u_2[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))]\\ &\qquad + \cdots\\ &\qquad + u_N[t, 0, \mu_1(x(0, t, x)),\dots, \mu_N(x(0, t, x))], \end{aligned}$$

where $u_1, u_2, \ldots, u_N$ are the components of the solution $U(t, 0, p_N)$ of the system (4.1.4). In view of (4.5.9) and the fact that $x_0 \in M_{(n-k)}$, using the relation (4.5.6) and the upper estimates in (4.5.8) and (4.5.11), we get

$$\begin{aligned}\sum_{i=1}^{N} V_i(t, x) &\leqslant \gamma_2\left[\sum_{i=1}^{N} \mu_i(x(0, t, x))\right] \\ &\leqslant \gamma_2[\alpha_2(\| x(0, t, x)\|)] \\ &\leqslant \gamma_2[\alpha_2(\beta_1^{-1}(\| x \|))] \\ &= a(\| x \|), \qquad a \in \mathscr{K}.\end{aligned}$$

Finally, as the solution $U_N^*(t)$ is nonnegative, we have

$$\sum_{i=1}^{N} V_i(t, x) \geqslant u_{NN}[t, 0, \mu_1(x(0, t, x)), \ldots, \mu_N(x(0, t, x))],$$

which, by using the inequality (4.5.7) and the lower estimates in (4.5.8) and (4.5.11), yields

$$\begin{aligned}\sum_{i=1}^{N} V_i(t, x) &\geqslant \gamma_1\left[\sum_{i=1}^{N} \mu_i(x(0, t, x))\right] \\ &\geqslant \gamma_1[\alpha_1(\| x(0, t, x)\|)] \\ &\geqslant \gamma_1[\alpha_1(\beta_2^{-1}(\| x \|))] \\ &= b(\| x \|), \qquad b \in \mathscr{K}.\end{aligned}$$

The proof is complete.

It is to be observed that the upper estimate in (4.5.5) and the inequality (4.5.6) ascertain the conditional stability of the null solutions of (4.1.1) and (4.1.4), respectively. The lower estimate in (4.5.5) and the estimate (4.5.7) are compatible with the conditional stability of the null solutions of (4.1.1) and (4.1.4), respectively.

THEOREM 4.5.2. Let assumptions (i) and (iii) of Theorem 4.5.1 hold. Suppose further that

(a) the solution $x(t, 0, x_0)$ of (4.1.1) satisfies the inequality

$$\begin{aligned}\beta_1(\| x_0 \|)\, \sigma_1(t) &\leqslant \| x(t, 0, x_0)\| \\ &\leqslant \beta_2(\| x_0 \|)\, \sigma_2(t), \qquad t \geqslant 0, \qquad x_0 \in M_{(n-k)}, \qquad (4.5.12)\end{aligned}$$

where $\beta_1, \beta_2 \in \mathscr{K}$ and $\sigma_1, \sigma_2 \in \mathscr{L}$;

(b) the solution $U(t, 0, p_N)$ of (4.1.4) verifies the estimate

$$\sum_{i=1}^{N} u_i(t, 0, p_N) \leqslant \gamma_2 \left(\sum_{i=1}^{N} u_{i0} \right) \delta_2(t), \qquad t \geqslant 0, \tag{4.5.13}$$

where $\gamma_2 \in \mathscr{K}$, $\delta_2 \in \mathscr{L}$, and, whenever $u_{i0} = 0$, $i = 1, 2,..., k$;

(c) the solution $U_N^*(t, 0, p_N)$ of (4.5.1) is such that

$$u_{NN}(t, 0, p_N) \geqslant \gamma_1 \left(\sum_{i=1}^{N} u_{i0} \right) \delta_1(t), \qquad t \geqslant 0, \tag{4.5.14}$$

where $\gamma_1 \in \mathscr{K}$, $\delta_1 \in \mathscr{L}$, and whenever $u_{i0} = 0$, $i = 1, 2,..., k$;

(d) $\gamma_1(r)$ is differentiable, and $\gamma_1'(r) \geqslant m \geqslant 0$;

(e) $\delta_1(t)$ and $\sigma_2(t)$ are such that $\delta_1(t) \geqslant m_1\sigma_2(t)$, $m_1 > 0$.

Then, there exists a function $V(t, x)$ with the properties (1), (2), (3) of Theorem 4.5.1 and

$$\begin{aligned} b(\| x \|) &\leqslant \sum_{i=1}^{N} V_i(t, x) \\ &\leqslant a(t, \| x \|), \qquad (t, x) \in J \times S_\rho, \end{aligned}$$

where $b \in \mathscr{K}$ and $a(t, r)$ belongs to class $\mathscr{K}$ for each fixed $t \in J$ and is continuous in t for each r.

Proof. Let $x(t, 0, x_0)$, $U(t, 0, p_N)$, and $U_N^*(t, 0, p_N)$ be the solutions of (4.1.1), (4.1.4), and (4.5.1) satisfying (4.5.12), (4.5.13), and (4.5.14), respectively. Choose any continuous function $\mu(x) \in R_+^N$ possessing continuous partial derivatives with respect to the components of x, such that (4.5.9) and

$$\beta_2(\| x \|) \leqslant \sum_{i=1}^{N} \mu_i(x) \leqslant \alpha(\| x \|), \qquad \beta_2, \alpha \in \mathscr{K}, \tag{4.5.15}$$

hold. Using the same definition (4.5.10) for $V(t, x)$ and proceeding as in Theorem 4.5.1, it can be easily shown that (1), (2), and (3) are valid.

Assumption (d) implies that

$$\gamma_1(r_1 r_2) \geqslant m r_1 r_2 . \tag{4.5.16}$$

The inequality (4.5.12), in view of the fact that $x = x(t, 0, x_0)$ and $x_0 = x(0, t, x)$, yields that

$$\beta_2^{-1} \left(\frac{\| x \|}{\sigma_2(t)} \right) \leqslant \| x(0, t, x) \| \leqslant \beta_1^{-1} \left(\frac{\| x \|}{\sigma_1(t)} \right), \tag{4.5.17}$$

where β_2^{-1}, β_1^{-1} both belong to class $\mathscr{K}$.

As in Theorem 4.5.1, using the definition (4.5.10) and the nonnegative character of $U_N^*(t)$, we get

$$\sum_{i=1}^{N} V_i(t, x) \geqslant u_{NN}[t, 0, \mu_1(x(0, t, x)),..., \mu_N(x(0, t, x))],$$

which, by virtue of (4.5.14), the lower estimates in (4.5.15) and (4.5.17), the relation (4.5.16), and the assumption (*e*), gives successively

$$\begin{aligned}\sum_{i=1}^{N} V_i(t, x) &\geqslant \gamma_1 \left[\sum_{i=1}^{N} \mu_i(x(0, t, x)) \right] \delta_1(t) \\ &\geqslant \gamma_1[\beta_2^{-1}(\| x(0, t, x)\|)]\, \delta_1(t) \\ &\geqslant \gamma_1 \left[\beta_2 \left(\beta_2^{-1} \left(\frac{\| x \|}{\sigma_2(t)}\right)\right)\right] \delta_1(t) \\ &\geqslant m m_1 \| x \| \\ &= b(\| x \|), \qquad b \in \mathscr{K}.\end{aligned}$$

Again, as before, making use of the definition of $V(t, x)$ and the relation (4.5.4) and (4.5.3), we obtain

$$\sum_{i=1}^{N} V_i(t, x) \leqslant \sum_{i=1}^{N} u_i(t, 0, \mu_1(x(0, t, x)),..., \mu_N(x(0, t, x))],$$

which, in its turn, allows the following estimates successively,

$$\begin{aligned}\sum_{i=1}^{N} V_i(t, x) &\leqslant \gamma_2 \left[\sum_{i=1}^{N} \mu_i(x(0, t, x)) \right] \delta_2(t) \\ &\leqslant \gamma_2[\alpha(\| x(0, t, x)\|)]\, \delta_2(t) \\ &\leqslant \gamma_2 \left[\alpha \left(\beta_1^{-1} \left(\frac{\| x \|}{\sigma_1(t)}\right)\right)\right] \delta_2(t) \\ &\equiv a(t, \| x \|),\end{aligned}$$

because of (4.5.13) and the upper estimates in (4.5.15) and (4.5.17). The theorem is proved.

Under the general assumptions of Theorem 4.5.2, it is not possible to prove the stronger requirement that $\sum_{i=1}^{N} V_i(t, x) \leqslant a(\| x \|)$. This can, however, be done if the estimates (4.5.12). (4.5.13), and (4.5.14) are modified as in the following:

THEOREM 4.5.3. Let assumptions (i) and (iii) of Theorem 4.5.1 hold. Assume that the inequalities (4.5.12), (4.5.13), and (4.5.14) in assumptions (a), (b), (c) of Theorem 4.5.2 are replaced by

$$\beta_1 \| x_0 \|^\alpha \sigma(t) \leqslant \| x(t, 0, x_0) \| \leqslant \beta_2 \| x_0 \|^\alpha \sigma(t), \qquad t \geqslant 0, \qquad x_0 \in M_{(n-k)}, \tag{4.5.18}$$

β_1, β_2, $\alpha > 0$ being constants, and $\sigma \in \mathscr{L}$;

$$\sum_{i=1}^{N} u_i(t, 0, p_N) \leqslant \gamma_2 \sum_{i=1}^{N} u_{i0}\, \delta(t), \qquad t \geqslant 0, \tag{4.5.19}$$

where $\gamma_2 > 0$ is a constant, $\delta \in \mathscr{L}$, and, whenever $u_{i0} = 0$, $i = 1, 2,..., k$;

$$u_{NN}(t, 0, p_N) \geqslant \gamma_1 \sum_{i=1}^{N} u_{i0}\, \delta(t), \qquad t \geqslant 0, \tag{4.5.20}$$

where $\gamma_1 > 0$ is a constant and $u_{i0} = 0$, $i = 1, 2,..., k$; respectively. Furthermore, let the functions $\delta(t)$ and $\sigma(t)$ be related by

$$\delta^\alpha(t) = \sigma^\beta(t),$$

for some constant $\beta > 0$. Then, there exists a function $V(t, x)$ with the properties (1), (2), (3) of Theorem 4.5.1, and

$$M_1 \| x \|^\rho \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant M_2 \| x \|^\rho,$$

where $M_1 = \gamma_1 \lambda_1 \beta_2^{-\rho}$, $M_2 = \gamma_2 \lambda_2 \beta_1^{-\rho}$, $\rho = \beta/\alpha$, and λ_1, λ_2 are some suitable positive constants.

Proof. By choosing the continuous function $\mu(x) \in R_+^N$ that satisfies (4.5.9) and

$$\lambda_1 \| x \|^\beta \leqslant \sum_{i=1}^{N} \mu_i(x) \leqslant \lambda_2 \| x \|^\beta,$$

λ_1, λ_2, β being constants greater than zero and following the proof of Theorem 4.5.2, with necessary changes, it is easy to construct the proof of the theorem.

It may be remarked that the conditional asymptotic stability of the null solutions (4.1.1) and (4.1.4) is expressed in terms of the upper estimate in (4.5.12) or (4.5.18) and (4.5.13) or (4.5.19), respectively. Also, the lower estimate in (4.5.12) or (4.5.18) and the inequality (4.5.14) or (4.5.20) are compatible with the conditional asymptotic stability of the trivial solutions of (4.1.1) and (4.1.4).

The conditional character of the stability notions in Theorems 4.5.1, 4.5.2, and 4.5.3 are due to the requirements that $x_0 \in M_{(n-k)}$ and $u_{i0} = 0$, $i = 1, 2, \ldots, k$. By dropping these conditions and modifying the technique suitably, it is easy to get a set of necessary conditions for the stability concepts, in terms of several Lyapunov functions.

4.6. Stability in tube-like domain

Lyapunov stability of the invariant set of a differential system does not rule out the possibility of asymptotic stability of the set, nor does the asymptotic stability of the invariant set guarantee any information about the rate of decay of the solution. Various definitions of stability and boundedness are, so to speak, one-sided estimates, and thus they are not strict concepts in a sense. It is natural to expect that an estimation of the lower bound for the rate at which the solutions approach the invariant set would yield interesting refinements of stability notions. We introduce below the concepts of strict stability and boundedness of solutions.

Let $Z(\alpha)$ and $\bar{Z}(\alpha)$ represent the sets

$$Z(\alpha) = [x \in S : \| x \| > \alpha],$$

$$\bar{Z}(\alpha) = [x \in S : \| x \| \geqslant \alpha],$$

respectively, and let $S(\alpha)$, $\bar{S}(\alpha)$, and $M_{(n-k)}$ have the same meaning as in Sect. 4.4. Let $x(t, t_0, x_0)$ be any solution of (4.1.1).

DEFINITION 4.6.1. The trivial solution of (4.1.1) is said to be

(CS_1) *conditionally strictly equistable* if, for any $\epsilon_1 > 0$, $t_0 \in J$, it is possible to find positive functions $\delta_1 = \delta_1(t_0, \epsilon_1)$, $\delta_2 = \delta_2(t_0, \epsilon_1)$, and $\epsilon_2 = \epsilon_2(t_0, \epsilon_1)$ that are continuous in t_0 for each ϵ_1, such that

$$\epsilon_2 < \delta_2 \leqslant \delta_1 < \epsilon_1,$$

$$x(t, t_0, x_0) \subset S(\epsilon_1) \cap Z(\epsilon_2), \qquad t \geqslant t_0,$$

provided

$$x_0 \in \bar{S}(\delta_1) \cap \bar{Z}(\delta_2) \cap M_{(n-k)};$$

(CS_2) *conditionally strictly uniformly stable* if δ_1, δ_2, and ϵ_2 in (CS_1) are independent of t_0;

(CS_3) *conditionally quasi-equi-asymptotically stable* if, given $\epsilon_1 > 0$, $\alpha_1 > 0$, and $t_0 \in J$, it is possible to find, for every α_2 satisfying $0 < \alpha_2 \leqslant \alpha_1$,

positive numbers ϵ_2, $T_1 = T_1(t_0, \epsilon_1, \alpha_1)$, and $T_2 = T_2(t_0, \epsilon_2, \alpha_2)$ such that

$$T_1 \leqslant T_2, \qquad \epsilon_2 < \epsilon_1, \qquad \epsilon_2 < \alpha_2,$$

$$x(t, t_0, x_0) \subset S(\epsilon_1) \cap Z(\epsilon_2), \qquad t_0 + T_1 \leqslant t \leqslant t_0 + T_2,$$

whenever

$$x_0 \in \bar{S}(\alpha_1) \cap \bar{Z}(\alpha_2) \cap M_{(n-k)};$$

(CS_4) *conditionally quasi-uniform-asymptotically stable* if T_1 and T_2 in (CS_3) are independent of t_0;

(CS_5) *conditionally attracting* if it is conditionally equistable and, in addition, (CS_3) holds;

(CS_6) *conditionally uniformly attracting* if it is conditionally uniformly stable and, in addition, (CS_4) holds.

The system (4.1.1) is said to be

(CS_7) *conditionally strictly equi-bounded* if, given $\alpha_1 > 0$, $t_0 \in J$, it is possible to find, for every α_2 satisfying $0 < \alpha_2 \leqslant \alpha_1$, positive functions $\beta_1 = \beta_1(t_0, \alpha_1)$, $\beta_2 = \beta_2(t_0, \alpha_1)$ that are continuous in t_0 for each α_1, such that

$$\beta_2 < \beta_1, \qquad \beta_2 < \alpha_2,$$

$$x(t, t_0, x_0) \subset S(\beta_1) \cap Z(\beta_2), \qquad t \geqslant t_0,$$

provided

$$x_0 \in \bar{S}(\alpha_1) \cap \bar{Z}(\alpha_2) \cap M_{(n-k)};$$

(CS_8) *conditionally strictly uniform bounded* if β_1, β_2 in (CS_7) are independent of t_0.

We observe that the foregoing notions assure that the motion remains in tube-like domains. In order to obtain the sufficient conditions for the stability of motion in tube-like domains, we have to estimate simultaneously both lower and upper bounds of the derivatives of Lyapunov functions and use the theory of differential inequalities. We are thus led to consider the two auxiliary systems

$$u' = g_1(t, u), \qquad u(t_0) = u_0 \geqslant 0, \tag{4.6.1}$$

$$v' = g_2(t, v), \qquad v(t_0) = v_0 \geqslant 0, \tag{4.6.2}$$

where $g_1, g_2 \in C[J \times R_+^N, R^N)$, $g_2(t, u) \leqslant g_1(t, u)$, and $g_1(t, u)$, $g_2(t, u)$ possess the quasi-monotone nondecreasing property in u for each $t \in J$. Then as a consequence of Corollary 1.7.1, we deduce that

$$\rho(t, t_0, v_0) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0,$$

provided

$$v_0 \leqslant u_0 ,$$

where $r(t, t_0, u_0)$, $\rho(t, t_0, v_0)$ are the maximal, minimal solutions of (4.6.1), (4.6.2), respectively.

Corresponding to definitions (CS_1)–(CS_8), we may formulate (CS_1^*)–(CS_8^*) with respect to the system (4.6.1) and (4.6.2). For example, (CS_1^*) would imply the following:

(CS_1^*) Given $\epsilon_1 > 0$, $t_0 \in J$, there exist positive functions $\delta_1 = \delta_1(t_0, \epsilon_1)$, $\delta_2 = \delta_2(t_0, \epsilon_1)$, $\epsilon_2 = \epsilon_2(t_0, \epsilon_1)$ that are continuous in t_0 for each ϵ_1 such that

$$\epsilon_2 < \delta_2 \leqslant \delta_1 < \epsilon_1 ,$$

$$\epsilon_2 < \sum_{i=1}^{N} \rho_i(t, t_0, v_0) \leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0) < \epsilon_1 , \qquad t \geqslant t_0 ,$$

if $u_{i0} = v_{i0} = 0$ $(i = 1, 2,..., k)$ and

$$\delta_2 \leqslant \sum_{i=1}^{N} v_{i0} \leqslant \sum_{i=1}^{N} u_{i0} \leqslant \delta_1 .$$

Let us restrict ourselves to proving conditional strict equistability only. Similar arguments with necessary modifications yield any desired result.

THEOREM 4.6.1. Assume that

(i) g_1, $g_2 \in C[J \times R_+^N, R^N]$, $g_2(t, u) \leqslant g_1(t, u)$, $g_1(t, 0) \equiv 0$, $g_2(t, 0) \equiv 0$, and $g_1(t, u)$, $g_2(t, u)$ possess the quasi-monotone nondecreasing property in u for each $t \in J$;

(ii) $V \in C[J \times S_\rho, R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, and, for $(t, x) \in J \times S_\rho$,

$$b(\| x \|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant a(\| x \|), \qquad a, b \in \mathscr{K};$$

(iii) $V_i(t, x) \equiv 0$ $(i = 1, 2,..., k)$, $k < n$, if $x \in M_{(n-k)}$, where $M_{(n-k)}$ is an $(n - k)$ dimensional manifold containing the origin;

(iv) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and, for $(t, x) \in J \times S_\rho$,

$$g_2(t, V(t, x)) \leqslant D^+V(t, x) \leqslant g_1(t, V(t, x)).$$

Then, if the auxiliary system satisfies condition (CS_1^*), the trivial solution of (4.1.1) is conditionally strictly equistable.

Proof. Let $0 < \epsilon_1 < \rho$ and $t_0 \in J$ be given. Assume that (CS_1^*) holds. Then, given $b(\epsilon_1) > 0$, $t_0 \in J$, there exist positive functions $\hat{\delta}_1 = \hat{\delta}_1(t_0, \epsilon_1)$, $\hat{\delta}_2 = \hat{\delta}_2(t_0, \epsilon_1)$, and $\hat{\epsilon}_2 = \hat{\epsilon}_2(t_0, \epsilon_1)$ such that

$$\hat{\epsilon}_2 < \hat{\delta}_2 \leqslant \hat{\delta}_1 < b(\epsilon_1),$$

$$\hat{\epsilon}_2 < \sum_{i=1}^{N} \rho_i(t, t_0, v_0)$$

$$\leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0) < b(\epsilon_1), \qquad t \geqslant t_0, \tag{4.6.3}$$

provided

$$\begin{gathered} u_{i0} = v_{i0} = 0 \qquad (i = 1, 2,..., k), \\ \hat{\delta}_2 \leqslant \sum_{i=1}^{N} v_{i0} = \sum_{i=1}^{N} u_{i0} \leqslant \hat{\delta}_1 . \end{gathered} \tag{4.6.4}$$

We choose $v_{i0} = V_i(t_0, x_0) = u_{i0}$ $(i = 1, 2,..., N)$ and $x \in M_{(n-k)}$ so that $v_{i0} = u_{i0} = 0$ $(i = 1, 2,..., k)$, by condition (iii). Let us make the following choice:

$$\delta_2 = b^{-1}(\hat{\delta}_2), \qquad \delta_1 = a^{-1}(\hat{\delta}_1), \qquad a(\epsilon_2) \leqslant \hat{\epsilon}_2, \qquad \epsilon_2 < \delta_2 .$$

Then, it is easy to verify that $\epsilon_2 < \delta_2 \leqslant \delta_1 < \epsilon_1$ and that ϵ_2, δ_2, δ_1 depend on t_0 and ϵ_1. Furthermore, $\delta_2 \leqslant \| x_0 \| \leqslant \delta_1$ implies

$$\hat{\delta}_2 \leqslant \sum_{i=1}^{N} V_i(t_0, x_0) \leqslant \hat{\delta}_1 ,$$

and vice versa. With this choice of ϵ_2, δ_2, and ϵ_1, the trivial solution of (4.1.1) is conditionally strictly equistable. Suppose that this is false. Then, there is a solution $x(t, t_0, x_0)$ of (4.1.1) satisfying

$$x_0 \in \bar{S}(\delta_1) \cap \bar{Z}(\delta_2) \cap M_{(n-k)}$$

such that, for some $t = t_1 > t_0$, it reaches the boundary of $S(\epsilon_1) \cap Z(\epsilon_2)$. This means that either $\| x(t_1, t_0, x_0)\| = \epsilon_1$ or $\| x(t_1, t_0, x_0)\| = \epsilon_2$. Also, $\| x(t, t_0, x_0)\| < \rho$, $t \in [t_0, t_1]$, and therefore, for $t \in [t_0, t_1]$, we can apply Theorem 4.1.3 to obtain

$$\rho(t, t_0, v_0) \leqslant V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0), \qquad t \in [t_0, t_1],$$

where $r(t, t_0, u_0)$, $\rho(t, t_0, v_0)$ are the maximal, minimal solutions of (4.6.1), (4.6.2), respectively, such that $v_0 = V(t_0, x_0) = u_0$. This implies that

$$\sum_{i=1}^{N} \rho_i(t, t_0, v_0) \leqslant \sum_{i=1}^{N} V_i(t, x(t, t_0, x_0))$$

$$\leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0), \tag{4.6.5}$$

for $t \in [t_0, t_1]$. In the first instance, if $\| x(t_1, t_0, x_0)\| = \epsilon_1$, using the right side inequality in (4.6.3) and (4.6.5), we arrive at the contradiction

$$b(\epsilon_1) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1, t_0, x_0))$$

$$\leqslant \sum_{i=1}^{N} r_i(t_1, t_0, u_0) < b(\epsilon_1).$$

On the other hand, if $\| x(t_1, t_0, x_0)\| = \epsilon_2$, we are led to a similar absurdity,

$$a(\epsilon_2) \geqslant \sum_{i=1}^{N} V_i(t_1, x(t_1, t_0, x_0))$$

$$\geqslant \sum_{i=1}^{N} \rho_i(t_1, t_0, v_0) > \hat{\epsilon}_2 \geqslant a(\epsilon_2),$$

because of the left side inequalities in (4.6.3) and (4.6.5). This shows that (CS_1) follows from (CS_1^*), and the proof of the theorem is complete.

4.7. Stability of asymptotically self-invariant sets

One has to consider, in many concrete problems like adaptive control systems, the stability of sets that are not self-invariant; this rules out Lyapunov stability, because those definitions of stability imply the existence of a self-invariant set. To describe such situations, the notion of eventual stability has been introduced in Sect. 3.14. It is easy to observe that, although such sets are not self-invariant in the usual sense, they are so in the asymptotic sense. This leads us to a new concept of asymptotically self-invariant sets. Evidently, asymptotically self-invariant sets form a special subclass of self-invariant sets, and therefore it is natural to expect that their stability properties closely resemble those of invariant sets.

Let $w \in C[R^n, R^m]$. Define

$$\| w(x)\| = \left[\sum_{i=1}^{m} w_i^2(x) \right]^{1/2}. \tag{4.7.1}$$

We shall denote the sets

$$[x \in R^n : \| w(x)\| = 0],$$
$$[x \in R^n : \| w(x)\| < \epsilon],$$
$$[x \in R^n : \| w(x)\| \leqslant \epsilon]$$

by G, $S(G, \epsilon)$, $\bar{S}(G, \epsilon)$, respectively. Suppose that $x(t) = x(t, t_0, x_0)$ is any solution of (4.1.1).

Definition 4.7.1. A set G is said to be *asymptotically self-invariant* with respect to the system (4.1.1) if, given any monotonic decreasing sequence $\{\epsilon_p\}$, $\epsilon_p \to 0$ as $p \to \infty$, there exists a monotonic increasing sequence $\{t_p(\epsilon)\}$, $t_p(\epsilon) \to \infty$ as $p \to \infty$, such that $x_0 \in G$, $t_0 \geqslant t_p(\epsilon)$, implies

$$x(t) \subset S(G, \epsilon_p), \qquad t \geqslant t_0, \qquad p = 1, 2, \ldots .$$

Let $M_{(n-k)}$ be an $(n - k)$ dimensional manifold containing the set G. We shall assume that G is an asymptotically self-invariant set with respect to the system (4.1.1).

Definition 4.7.2. The asymptotically self-invariant set G of the system (4.1.1) is said to be (AS_1) *conditionally equistable* if, for each $\epsilon > 0$, there exists a $t_1(\epsilon)$, $t_1(\epsilon) \to \infty$ as $\epsilon \to 0$, and a $\delta = \delta(t_0, \epsilon)$, $t_0 \geqslant t_1(\epsilon)$, which is continuous in t_0 for each ϵ such that

$$x(t) \subset S(G, \epsilon), \qquad t \geqslant t_0 \geqslant t_1(\epsilon),$$

provided

$$x_0 \in \bar{S}(G, \delta) \cap M_{(n-k)} .$$

On the basis of this definition, it is easy to formulate the remaining notions (AS_2)–(AS_8) corresponding to (C_1)–(C_8) of Sect. 4.4.

The following theorem gives sufficient conditions for the set G to be asymptotically self-invariant with respect to the system (4.1.1).

Theorem 4.7.1. Assume that

(i) $g \in C[J \times R_+^N, R^N]$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(ii) $V \in C[J \times S(G, \rho), R_+^N]$, $V(t, x)$ is locally Lipschitzian in x,

$$b(\| w(x)\|) \leqslant \sum_{i=1}^{N} V_i(t, x), \qquad (t, x) \in J \times S(G, \rho), \qquad b \in \mathscr{K}, \tag{4.7.2}$$

and

$$\sum_{i=1}^{N} V_i(t, x) = \sigma(t) \qquad \text{if} \quad x \in G, \tag{4.7.3}$$

where $\sigma \in \mathscr{L}$;

(iii) $V_i(t, x) \equiv 0$ $(i = 1, 2,..., k)$, $k < n$, if $x \in M_{(n-k)}$, where $M_{(n-k)}$ is an $(n - k)$ dimensional manifold containing the set G;

(iv) $f \in C[J \times S(G, \rho), R^n]$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times S(G, \rho);$$

(v) for any function $\beta(t, u)$, which is continuous for $t \geqslant 0$, $u \geqslant 0$, decreasing in t for each fixed u, increasing in u for each fixed t such that

$$\lim_{t \to \infty} \lim_{u \to 0} \beta(t, u) = 0, \tag{4.7.4}$$

we have

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) \leqslant \beta\left(t_0, \sum_{i=1}^{N} u_{i0}\right), \qquad t \geqslant t_0 \geqslant 0, \tag{4.7.5}$$

provided $u_{i0} = 0$ $(i = 1, 2,..., k)$, where $u(t, t_0, u_0)$ is any solution of (4.1.4).

Then, the set $G = [x \in R^n : \| w(x)\| = 0]$ is asymptotically self-invariant with respect to (4.1.1).

Proof. Let $x_0 \in G$. Since $G \subset M_{(n-k)}$, it follows that $x_0 \in M_{(n-k)}$. As a consequence, we have, by (iii), $V_i(t_0, x_0) \equiv 0$ $(i = 1, 2,..., k)$, $k < n$. We choose $u_{i0} = V_i(t_0, x_0)$ $(i = 1, 2,..., N)$. Then, because of (4.7.3), we obtain

$$\sum_{i=1}^{N} u_{i0} = \sum_{i=1}^{N} V_i(t_0, x_0) = \sigma(t_0). \tag{4.7.6}$$

Consider the function $\gamma(t) = \beta(t, \sigma(t))$, which decreases to zero as $t \to \infty$ because of the assured monotonic properties of the functions β and σ. Let now $\{\epsilon_p\}$ be a decreasing sequence such that $\epsilon_p \to 0$ as $p \to \infty$. Then, the sequence $\{b(\epsilon_p)\}$ is a similar sequence. Since $\gamma(t_0) \to 0$ as $t_0 \to \infty$, it is possible to find an increasing sequence $\{t_p(\epsilon)\}$, $t_p(\epsilon) \to \infty$ as $p \to \infty$, such that

$$\gamma(t_0) < b(\epsilon_p), \qquad t_0 \geqslant t_p(\epsilon), \qquad p = 1, 2,... . \tag{4.7.7}$$

We claim that $x_0 \in G$ implies that $x(t) \subset S(G, \epsilon_p)$, $t \geqslant t_0 \geqslant t_p(\epsilon)$, for each $p = 1, 2,\ldots$. Suppose, on the contrary, that there exists a solution $x(t)$ of (4.1.1) such that $x_0 \in G$, $t_0 \geqslant t_p(\epsilon)$ for a certain p, $\| w(x(t_1))\| = \epsilon_p$ for some $t = t_1 > t_0 \geqslant t_p(\epsilon)$, and

$$\| w(x(t))\| \leqslant \epsilon_p < \rho, \qquad t \in [t_0, t_1].$$

For $t \in [t_0, t_1]$, we obtain, on account of Theorem 4.1.1, the inequality

$$\sum_{i=1}^{N} V_i(t, x(t)) \leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0), \tag{4.7.8}$$

where $r(t, t_0, u_0)$ is the maximal solution of (4.1.4). At $t = t_1$, we arrive at the contradiction

$$\begin{aligned} b(\epsilon_p) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)) &\leqslant \sum_{i=1}^{N} r_i(t_1, t_0, u_0) \\ &\leqslant \beta(t_0, \sigma(t_0)) \\ &= \gamma(t_0) < b(\epsilon_p), \end{aligned}$$

making use of the relations (4.7.5), (4.7.6), (4.7.7), and (4.7.8). This proves that the set G is asymptotically self-invariant with respect to the system (4.1.1).

If we assume that the set $u = 0$ is asymptotically self-invariant with respect to the auxiliary system (4.1.4), we have the following definition parallel to Definition 4.7.2.

Definition 4.7.3. The asymptotically self-invariant set $u = 0$ of the system (4.1.4) is said to be (AC_1^*) *conditionally equi-stable* if, for each $\epsilon > 0$, there exists a $t_1(\epsilon)$, $t_1(\epsilon) \to \infty$ as $\epsilon \to 0$, and a $\delta = \delta(t_0, \epsilon)$, $t_0 \geqslant t_1(\epsilon)$, which is continuous in t_0 for each ϵ, such that

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < \epsilon, \qquad t \geqslant t_0 \geqslant t_1(\epsilon),$$

provided

$$\sum_{i=1}^{N} u_{i0} \leqslant \delta, \qquad u_{i0} = 0 \qquad (i = 1, 2,\ldots, k).$$

The following theorem assures the conditional equistability of the asymptotically self-invariant set G.

THEOREM 4.7.2. Suppose that hypotheses (i), (ii), (iii), and (iv) of Theorem 4.7.1 hold, except (4.7.3). Assume further that the set G is asymptotically self-invariant and

$$\sum_{i=1}^{N} V_i(t, x) \leqslant a(t, \| w(x)\|), \qquad (t, x) \in J \times S(G, \rho), \tag{4.7.9}$$

where the function $a(t, r)$ is defined and continuous for $t \geqslant 0$, $r \geqslant 0$, monotonic decreasing in t for each fixed r, monotonic increasing in r for each fixed t, and

$$\lim_{t\to\infty} \lim_{r\to 0} a(t, r) = 0.$$

Then (AC_1^*) implies (AC_1).

Proof. Let $0 < \epsilon < \rho$ be given. Assume that the definition (AC_1^*) holds. Then, given $b(\epsilon) > 0$, there exist a $t_1(\epsilon)$, $t_1(\epsilon) \to \infty$ as $\epsilon \to 0$, and a $\delta = \delta(t_0, \epsilon)$, $t_0 \geqslant t_1(\epsilon)$ such that

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\epsilon), \qquad t \geqslant t_0 \geqslant t_1(\epsilon), \tag{4.7.10}$$

provided

$$\sum_{i=1}^{N} u_{i0} \leqslant \delta, \qquad u_{i0} = 0 \qquad (i = 1, 2,..., k). \tag{4.7.11}$$

Choose $u_{i0} = V_i(t_0, x_0)$, $i = 1, 2,..., k$, and $x_0 \in M_{(n-k)}$ so that $u_{i0} = 0$ $(i = 1, 2,..., k)$, by condition (iii). If we now make the choice that $\sum_{i=1}^{N} u_{i0} = a(t_0, \| w(x_0)\|)$, the assumptions on $a(t, r)$ imply the existence of positive numbers $t_2(\epsilon)$ and $\delta_1 = \delta_1(t_0, \epsilon)$, $t_0 \geqslant t_2(\epsilon)$, such that

$$a(t_0, \| w(x_0)\|) \leqslant \delta, \qquad \| w(x_0)\| \leqslant \delta_1, \tag{4.7.12}$$

provided $t_0 \geqslant t_2(\epsilon)$. Let $t_3(\epsilon) = \max[t_1(\epsilon), t_2(\epsilon)]$. It can then be claimed that, if $x_0 \in \bar{S}(G, \delta_1) \cap M_{(n-k)}$, we have $x(t, t_0, x_0) \subset S(G, \epsilon)$ for $t \geqslant t_0 \geqslant t_3(\epsilon)$, where $x(t, t_0, x_0)$ is any solution of (4.1.1). Let us assume that this is not true. Then, there exists a solution $x(t)$ of (4.1.1) such that, whenever $x_0 \in \bar{S}(G, \delta_1) \cap M_{(n-k)}$, $x(t) \subset S(G, \epsilon)$ for $t \in [t_0, t_1]$, $t_1 > t_0 \geqslant t_3(\epsilon)$, and $x(t_1)$ lies on the boundary of $S(G, \epsilon)$. This implies that

$$\| w(x(t))\| \leqslant \epsilon, \qquad t \in [t_0, t_1],$$

and $\| w(x(t_1))\| = \epsilon$. Thus, there results

$$b(\epsilon) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)). \tag{4.7.13}$$

Furthermore, for $t \in [t_0, t_1]$, we obtain the inequality

$$V(t, x(t)) \leqslant r(t, t_0, u_0), \tag{4.7.14}$$

in view of Theorem 4.1.1, $r(t, t_0, u_0)$ being the maximal solution of (4.1.4). Since the choice of u_{i0} and the relation (4.7.12) guarantee that, whenever $x_0 \in \bar{S}(G, \delta_1) \cap M_{(n-k)}$, the condition (4.7.11) is satisfied, it is easy to derive, from (4.7.10) and (4.7.14), the inequality

$$\sum_{i=1}^{N} V_i(t_1, x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0, u_0) < b(\epsilon).$$

This relation is incompatible with (4.7.13), thereby establishing (AC_1).

Corollary 4.7.1. Under the assumptions of Theorem 4.7.2, the conditional equistability of the trivial solution of (4.1.4) assures the definition (AC_1).

We can easily prove the statements corresponding to the definitions (AC_2)–(AC_8), on the basis of Theorem 4.7.2. To show the close relationship between theorems of this section and Sect. 3.14, we shall merely state below a theorem parallel to Theorem 3.14.1.

Theorem 4.7.3. Assume that

(i) $g \in C[J \times R_+^N, R^N]$, $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$, and the asymptotically self-invariant set $u = 0$ of (4.1.4) is conditionally uniformly stable;

(ii) $V \in C[J \times S(G, \rho), R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, and

$$b(\| w(x)\|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant a(\| w(x)\|),$$

for $0 < r < \| w(x)\| < \rho$ and $t \geqslant \theta(r)$, where $a, b \in \mathscr{K}$ and the function $\theta(r) \geqslant 0$ is monotonic decreasing in r for $0 < r < \rho$;

(iii) $V_i(t, x) = 0$ $(i = 1, 2,..., k)$, $k < n$ if $x \in M_{(n-k)}$, where $M_{(n-k)}$ is an $(n - k)$ dimensional manifold containing the set G;

(iv) $f \in C[J \times S(G, \rho), R^n]$, the set G is asymptotically self-invariant with respect to the system (4.1.1), and

$$D^+V(t, x) \leqslant g(t, V(t, x)),$$

for $0 < r < \| w(x)\| < \rho$ and $t \geqslant \theta(r)$.

Then, the asymptotically self-invariant set G is conditionally uniformly stable.

Analogous to the boundedness concepts (B_1)–(B_8) defined in Sect. 3.13, we have the following weaker notions.

DEFINITION 4.7.4. The system (4.1.1) is said to be, with respect to the set G, (EB_1) *conditionally eventually equi-bounded* if, given $\alpha \geqslant 0$, there exist $t_1(\alpha) \geqslant 0$ and $\beta = \beta(t_0, \alpha)$, $t_0 \geqslant t_1(\alpha)$, which is continuous in t_0 for each α, such that

$$x(t) \subset S(G, \beta), \qquad t \geqslant t_0 \geqslant t_1(\alpha),$$

provided

$$x_0 \in \bar{S}(G, \alpha) \cap M_{(n-k)} .$$

The remaining definitions (EB_2)–(EB_8) may be easily formulated. As previously, the definitions (EB_1^*)–(EB_8^*) refer to the conditional boundedness concepts with respect to the system (4.1.4). A typical theorem on eventual boundedness is the following:

THEOREM 4.7.4. Suppose that

(i) $g \in C[J \times R_+^N, R_+^N]$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(ii) $V \in C[J \times R^n, R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, and

$$b(\| w(x)\|) \leqslant \sum_{i=1}^{N} V_i(t, x)$$

$$\leqslant a(t, \| w(x)\|), \qquad t \geqslant 0, \qquad x \in R^n,$$

where $a(t, r)$ is continuous for $t \geqslant 0$, $r \geqslant 0$, montonic decreasing in t for each r, monotonic increasing in r for each t, and

$$\lim_{t \to \infty} \lim_{r \to 0} a(t, r) = 0,$$

and $b \in \mathscr{K}$ on the interval $0 \leqslant r < \infty$ such that $b(r) \to \infty$ as $r \to \infty$;

(iii) $V_i(t, x) \equiv 0$ $(i = 1, 2,..., k)$, $k < n$, if $x \in M_{(n-k)}$;

(iv) $f \in C[J \times R^n, R^n]$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times R^n.$$

Then the condition (EB_1^*) implies (EB_1).

Proof. Let $\alpha \geqslant 0$ be given. Suppose that $x_0 \in \bar{S}(G, \alpha) \cap M_{(n-k)}$. Because of the assumptions on $a(t, r)$, it is possible to find two positive numbers $\gamma = \gamma(\alpha)$ and $t_1(\alpha)$ such that

$$a(t_0, \alpha) \leqslant \gamma \qquad \text{if} \quad t_0 \geqslant t_1(\alpha). \tag{4.7.15}$$

Assume that (EB_1^*) holds. Then, given $\gamma > 0$, there exist two numbers $t_2(\alpha)$ and $\beta = \beta(t_0, \alpha)$, $t_0 \geqslant t_2(\alpha)$, such that

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < \beta, \qquad t \geqslant t_0 \geqslant t_2(\alpha), \tag{4.7.16}$$

whenever

$$\sum_{i=1}^{N} u_{i0} \leqslant \gamma, \qquad u_{i0} = 0 \qquad (i = 1, 2,..., k). \tag{4.7.17}$$

Let $t_3(\alpha) = \max[t_1(\alpha), t_2(\alpha)]$. Choose $u_{i0} = V_i(t_0, x_0)$, $t_0 \geqslant t_3(\alpha)$ $(i = 1, 2,..., k)$, and $\sum_{i=1}^{N} u_{i0} = a(t_0, \| w(x_0)\|)$. Since $x_0 \in \bar{S}(G, \alpha) \cap M_{(n-k)}$, this implies, in view of condition (iii), that $u_{i0} = 0$ $(i = 1, 2,..., k)$. Moreover, the condition (4.7.17) is satisfied, in view of this choice, and consequently (4.7.16) is true. Since $b(r) \to \infty$ as $r \to \infty$, there exists a $\beta_1 = \beta_1(t_0, \alpha)$ such that

$$b(\beta_1) \geqslant \beta. \tag{4.7.18}$$

We can now conclude that (EB_1) holds with β_1 and $t_3(\alpha)$. The assumption that this is false leads to the existence of a $t_1 > t_0 \geqslant t_3(\alpha)$ and a solution $x(t)$ with $x_0 \in \bar{S}(G, \alpha) \cap M_{(n-k)}$, such that

$$\| w(x(t_1))\| = \beta_1$$

at $t = t_1 > t_0 \geqslant t_3(\alpha)$. By assumption (iv) and Theorem 4.1.1, we can infer that

$$\sum_{i=1}^{N} V_i(t, x(t)) \leqslant \sum_{i=1}^{N} r_i(t, t_0, u_0), \qquad t \geqslant t_0 \geqslant t_3(\alpha),$$

which, because of the relation (4.7.16) and assumption (ii), shows that

$$b(\beta_1) \leqslant \sum_{i=1}^{N} V_i(t_1, x(t_1)) \leqslant \sum_{i=1}^{N} r_i(t_1, t_0, u_0) < \beta.$$

This is a contradiction to the choice of β_1 in (4.7.18), and hence we claim that (EB_1) holds. The proof is complete.

4.8. Stability of conditionally invariant sets

We shall introduce in this section the concept of a conditionally invariant set with respect to a given set and consider the stability properties of such sets.

DEFINITION 4.8.1. Let A and B be any two subsets of R^n such that $A \subset B$. Then, the set B is said to be *conditionally invariant* with respect to the set A for the differential system (4.1.1) if $x_0 \in A$ implies that $x(t, t_0, x_0) \subset B$ for all $t \geqslant t_0 \geqslant 0$.

Let $w \in C[R^n, R^m]$, and let $\| w(x)\|$ mean the same norm of w defined by (4.7.1). Let us continue to use the sets

$$G = [x \in R^n : \| w(x)\| = 0],$$

$$S(G, \epsilon) = [x \in R^n : \| w(x)\| < \epsilon],$$

$$\bar{S}(G, \epsilon) = [x \in R^n : \| w(x)\| \leqslant \epsilon],$$

and let us designate the set $\bar{S}(G, \alpha)$ by B. Suppose that the set $B = \bar{S}(G, \alpha)$ is conditionally invariant with respect to G, for some $\alpha > 0$. Let $M_{(n-k)}$ denote, as before, an $(n - k)$ dimensional manifold containing the set G. We define

$$S(B, \epsilon) = S(G, \alpha + \epsilon), \qquad \epsilon > 0.$$

DEFINITION 4.8.2. The conditionally invariant set B with respect to the set G and the system (4.1.1) is said to be (CC_1) *conditionally equistable* if, for each $\epsilon > 0$ and $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$, which is continuous in t_0 for each ϵ, such that

$$x(t, t_0, x_0) \subset S(B, \epsilon), \qquad t \geqslant t_0$$

whenever

$$x_0 \in \bar{S}(G, \delta) \cap M_{(n-k)} .$$

Evidently, on the strength of (CC_1), we can define (CC_2)–(CC_8) corresponding to (C_2)–(C_8).

REMARK 4.8.1. We observe that the set B need not be self-invariant. If $\alpha = 0$, these definitions coincide with (C_1)–(C_8), that is, the conditional stability concepts of the self-invariant G.

To define the corresponding definitions (CC_1^*)–(CC_8^*) for the auxiliary system (4.1.4), let us define the set, for some $\beta > 0$,

$$B^* = \left[u \in R_+^N \colon \sum_{i=1}^{N} u_i \leqslant \beta\right], \tag{4.8.1}$$

and assume that B^* is conditionally invariant with respect to the set $u = 0$ and the system (4.1.4).

Definition 4.8.3. The conditionally invariant set B^* with respect to the set $u = 0$ and the system (4.1.4) is said to be (CC_1^*) *conditionally equistable* if, for each $\epsilon > 0$ and $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$, which is continuous in t_0 for each ϵ, such that

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < \beta + \epsilon, \qquad t \geqslant t_0,$$

provided

$$\sum_{i=1}^{N} u_{i0} < \delta, \qquad u_{i0} = 0 \qquad (i = 1, 2,..., k).$$

Theorem 4.8.1. Assume that

(i) $g \in C[J \times R_+^N, R^N]$, $g(t, 0) \equiv 0$, and $g(t, u)$ is quasi-monotone nondecreasing in u for each $t \in J$;

(ii) $V \in C[J \times R^n, R_+^N]$, $V(t, x)$ is locally Lipschitzian in x, and

$$b(\| w(x)\|) \leqslant \sum_{i=1}^{N} V_i(t, x) \leqslant a(\| w(x)\|), \qquad (t, x) \in J \times R^n,$$

where $a, b \in \mathscr{K}$ on the interval $[0, \infty)$ and

$$b(r) \to \infty \qquad \text{as} \quad r \to \infty;$$

(iii) $f \in C[J \times R^n, R^n]$, and

$$D^+V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in J \times R^n.$$

Then, if the set B^* is conditionally invariant with respect to the set $u = 0$ and the system (4.1.4), the set $B = \bar{S}(G, \alpha)$, where $\alpha = b^{-1}(\beta)$, is conditionally invariant with respect to the set G and the system (4.1.1).

Proof. Assume that the set B^* defined by (4.8.1) is a conditionally invariant set. This implies that, if $u_{i0} = 0$ $(i = 1, 2,..., N)$,

$$\sum_{i=1}^{N} u_i(t, t_0, 0) \leqslant \beta, \qquad t \geqslant t_0 \geqslant 0. \tag{4.8.2}$$

Let us choose $u_{i0} = V_i(t_0, x_0)$ $(i = 1, 2,..., N)$. Then, it follows that $x_0 \in G$ and $V_i(t_0, x_0) = 0$ $(i = 1, 2,..., N)$ hold simultaneously. By Theorem 4.1.1, we obtain

$$\sum_{i=1}^{N} V_i(t, x(t, t_0, x_0)) \leqslant \sum_{i=1}^{N} r_i(t, t_0, 0), \qquad t \geqslant t_0, \tag{4.8.3}$$

where $r(t, t_0, u_0)$ is the maximal solution of (4.1.4) through (t_0, u_0). Since $b(\| w(x)\|) \leqslant \sum_{i=1}^{N} V_i(t, x)$, we readily get the inequality

$$b(\| w(x(t, t_0, x_0))\|) \leqslant \beta, \qquad t \geqslant t_0,$$

in view of (4.8.2) and (4.8.3). As a consequence, we deduce that, if $x_0 \in G$, $x(t, t_0, x_0) \subset \bar{S}(G, \alpha)$, $t \geqslant t_0$, where $\alpha = b^{-1}(\beta)$. The conditional invariancy of the set B is immediate, and the proof is complete.

REMARK 4.8.2. Notice that the β occurring in (4.8.2) may depend on t_0, in which case α depends on t_0, and, as a result, the set B depends on t_0. This suggests that the invariant sets we generally consider are, in a sense, uniform invariant sets, and perhaps a classification of invariant sets and the study of their stability properties may be of some interest.

Regarding the stability behavior of the conditionally invariant set B, we have the following:

THEOREM 4.8.2. Assume that conditions (i), (ii), and (iii) of Theorem 4.8.1 hold. Suppose further that $V_i(t, x) = 0$ $(i = 1, 2,..., k)$, $k < n$, if $x \in M_{(n-k)}$. Then, if one of the conditions (CC_1^*)–(CC_8^*) is satisfied, the corresponding one of the conditions (CC_1)–(CC_8) is assured.

Proof. We shall only indicate the proof corresponding to the statement (CC_8), that is, the conditional quasi-uniform asymptotic stability of the conditional invariant set B.

Let $\epsilon > 0$, $\gamma > 0$, and $t_0 \in J$ be given. Suppose that

$$x_0 \in \bar{S}(G, \gamma) \cap M_{(n-k)},$$

so that we can infer that $V_i(t_0, x_0) = 0$ $(i = 1, 2,..., k)$. Choose $u_{i0} = V_i(t_0, x_0)$ $(i = 1, 2,..., N)$. Then, we have by Theorem 4.1.2 that every solution $x(t, t_0, x_0)$ of (4.1.1) exists for $t \geqslant t_0$ and satisfies

$$V(t, x(t, t_0, x_0)) \leqslant r(t, t_0, u_0), \qquad t \geqslant t_0, \tag{4.8.4}$$

where $r(t, t_0, u_0)$ is the maximal solution of (4.1.4). Define $\gamma_1 = a(\gamma)$,

and assume that (CC_8^*) holds. Let $\alpha = b^{-1}(\beta)$. Then, given $b(\alpha + \epsilon) > 0$, $\gamma_1 > 0$, and $t_0 \in J$, there exists a positive number $T = T(\gamma, \epsilon)$ such that

$$\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\alpha + \epsilon), \qquad t \geq t_0 + T, \tag{4.8.5}$$

provided

$$\sum_{i=1}^{N} u_{i0} \leq \gamma_1, \qquad u_{i0} = 0 \qquad (i = 1, 2,..., k). \tag{4.8.6}$$

Clearly, by the choice of γ_1 and u_{i0}, the condition (4.8.6) is satisfied. Hence, we obtain, using (4.8.4), (4.8.5), and the fact that

$$b(\| w(x)\|) \leq \sum_{i=1}^{N} V_i(t, x),$$

the relation

$$b(\| w(x(t, t_0, x_0))\|) < b(\alpha + \epsilon), \qquad t \geq t_0 + T,$$

whenever $x_0 \in \bar{S}(G, \gamma) \cap M_{(n-k)}$. Evidently, this implies that the conditionally invariant set B is conditionally quasi-uniform asymptotically stable. The proof of the theorem is thus complete.

4.9. Existence and stability of stationary points

This section is concerned with the conditions sufficient to assure the existence of y_0 satisfying

$$f(y_0) = 0 \tag{4.9.1}$$

and the stability of the solution $x(t) \equiv y_0$ of the autonomous differential system

$$x' = f(x), \qquad x(0) = x_0, \tag{4.9.2}$$

where $f \in C[R^n, R^n]$.

THEOREM 4.9.1. Assume that

(i) $V \in C[R^n, R_+^N]$, $V(x)$ is locally Lipschitzian in x, and

$$\sum_{i=1}^{N} V_i(x) \to \infty \qquad \text{as} \quad \| x \| \to \infty;$$

(ii) $g \in C[R_+^N, R^N]$, $g(u)$ is quasi-monotone nondecreasing in u, and $D^+V(x) \leq g(V(x))$, $x \in R^n$;

(iii) $Q \in C[R_+^N, R_+]$, $Q(v)$ is monotone nondecreasing in v, and $Q(v(x)) = 0$ only if $f(x) = 0$;

(iv) for a certain u_0, the system

$$u' = g(u), \qquad u(0) = u_0 > 0 \tag{4.9.3}$$

possesses the maximal solution $r(t, 0, u_0)$ defined for $0 \leqslant t < \infty$ such that $r(t, 0, u_0)$ is bounded and satisfies

$$Q(r(t, 0, u_0)) \to 0 \qquad \text{as} \quad t \to \infty. \tag{4.9.4}$$

Then, if $x(t)$ is any solution of (4.9.2), it exists and is bounded for $t \in J$, and every cluster point (ω-limit point) y_0 of $x(t)$ satisfies (4.9.1).

Proof. Let $x(t)$ be a solution of (4.9.2). Then, by Theorem 4.1.2, $x(t)$ exists for $0 \leqslant t < \infty$. Furthermore, if $V(t_0, x_0) \leqslant u_0$,

$$V(x(t)) \leqslant r(t, 0, u_0), \qquad t \geqslant 0, \tag{4.9.5}$$

where $r(t, 0, u_0)$ is the maximal solution of (4.9.3). The assumptions that $r(t, 0, u_0)$ is bounded for $t \geqslant 0$ implies, in view of (i), the boundedness of $x(t)$. Also, the function Q being monotonic nondecreasing, we have, by (4.9.5),

$$Q(V(x(t))) \leqslant Q(r(t, 0, u_0)),$$

which, on account of (4.9.4), guarantees that

$$\lim_{t \to \infty} Q(V(x(t))) = 0.$$

Hence, every ω-limit point y_0 of $x(t)$ satisfies

$$Q(V(y_0)) = 0,$$

and (4.9.1) follows, because of assumption (iii). This proves the theorem.

COROLLARY 4.9.1. Let the hypotheses of Theorem 4.9.1 hold, except that $Q(V(x)) = 0$ only if $f(x) = 0$ is replaced by

$$Q(V(x)) = 0 \qquad \text{only if} \quad U(x) = 0,$$

where $U \in C[R^n, R^n]$. Then the assertion of Theorem 4.9.1 remains valid if (4.9.1) is replaced by

$$U(y_0) = 0.$$

THEOREM 4.9.2. Suppose that the conditions of Theorem 4.9.1 are satisfied. In addition, assume that, for every $u_0 > 0$, the maximal solution $r(t, 0, u_0)$ of (4.9.3) exists on $0 < t < \infty$ and is uniformly bounded for $t \geqslant 0$ and bounded u_0 and satisfies (4.9.4) uniformly for bounded u_0. Then the set

$$Z = [x : f(x) = 0]$$

is nonempty and connected.

Proof. It is clear that, under the assumptions of the theorem, every solution $x(t)$ of (4.9.2) exists for $0 \leqslant t < \infty$, and, given any $\alpha > 0$, there exists a $\beta(\alpha)$ such that

$$\| x(t)\| \leqslant \beta(\alpha), \qquad t \geqslant 0,$$

provided $\| x_0 \| \leqslant \alpha$. Furthermore, every ω-limit point y_0 satisfies (4.9.1). By Theorem 4.9.1, it follows that the set Z is nonempty. Hence, only connectedness remains to be proved.

Let ϵ, α be arbitrarily positive numbers. Then, it follows from (ii) that there exists a $\delta = \delta(\epsilon, \alpha) > 0$ such that

$$Q(V(x)) \geqslant \delta \qquad \text{if} \quad d(x, Z) \geqslant \epsilon, \| x \| \leqslant \beta(\alpha).$$

Hence, by the uniformity of (4.9.4) and by (4.9.5), it is possible to find a $T = T(\epsilon, \alpha)$ such that

$$d(x(t), Z) < \epsilon \qquad \text{if} \quad t \geqslant T, \| x_0 \| \leqslant \alpha. \tag{4.9.6}$$

Since all solutions $x(t)$ of (4.9.2), for arbitrary x_0, exist on $0 \leqslant t < \infty$, it follows by a generalization of H. Kneser's theorem that the set $Z_\alpha(t)$ of points z reached by some solution of (4.9.2) at a time $t \geqslant 0$, when $\| x_0 \| \leqslant \alpha$, that is,

$$Z_\alpha(t) = [z : z = x(t), \| x_0 \| \leqslant \alpha],$$

is closed and connected. We notice that the set

$$Z_\alpha = Z \cap [x : \| x \| \leqslant \alpha]$$

is contained in $Z_\alpha(t)$ for $t \geqslant 0$ [for, if $y_0 \in Z$, then $x(t) \equiv y_0$ is a solution of (4.9.2)].

Let x_a, x_b be two arbitrary points of $Z_\alpha \subset Z_\alpha(T)$. Then, there exists a finite set of points $x_a = x_0, x_1, \ldots, x_{N+1} = x_b$ in $Z_\alpha(T)$ such that $\| x_i - x_{i+1} \| < \epsilon$, if $i = 0, 1, \ldots, N$. In view of (4.9.6), there is a point

$x^i \in Z$ satisfying $\| x_i - x^i \| < \epsilon$ for $i = 1, 2, \ldots, N$. Hence, there is a finite set of points $x_a = x_0 = x^0, x^1, \ldots, x^N, x^{N+1} = x_{N+1} = x_b$ in Z such that $\| x^i - x^{i+1} \| < 3\epsilon$, for $i = 0, 1, \ldots, N$. Since ϵ is arbitrary and $Z = \cap Z_\alpha$, $\alpha \geqslant 0$, the set Z is connected. This proves the theorem.

COROLLARY 4.9.2. Assume that the conditions of Theorem 4.9.2 hold. Suppose that the zeros of $f(x)$ are isolated. Then $f(x)$ has a unique zero y_0, and the solution $x(t) \equiv y_0$ of (4.9.2) is completely uniformly asymptotically stable.

4.10. Notes

Section 4.1 introduces comparison theorems that are useful when several Lyapunov functions are employed (see Lakshmikantham [13]).

The results of Sects. 4.2 and 4.3 have been taken from the work of Matrosov [1]. For the results contained in Sect. 4.4, see Lakshmikantham [13] and Matrosov [2].

Converse theorems of Sect. 4.5 are due to Lakshmikantham *et al.* [1], whereas the results of Sect. 4.6 concerning the stability in tube-like domains are taken from the work of Charlu *et al.* [1].

The notion of asymptotically self-invariant sets is introduced by Lakshmikantham and Leela [1], and the contents of Sect. 4.7 are based on their work.

The results of Sect. 4.8 dealing with the criteria for the stability of conditionally invariant sets are due to Kayande and Lakshmikantham [1].

Section 4.9 deals with the results due to Hartman [6].

For related work using several Lyapunov functions, see Antosiewicz [4], Bellman [4], D'Ambrosio and Lakshmikantham [1], and Lakshmikantham and Verma [1].

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VOLTERRA INTEGRAL EQUATIONS

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Chapter 5

5.0. Introduction

We treat in this chapter certain problems concerning Volterra integral equations. We consider successively basic integral inequalities, local and global existence theorems, the existence of extremal solutions, uniqueness of solutions, bounds, and error estimates of approximate solutions. We discuss the asymptotic behavior of solutions by suitably choosing Lyapunov-like functions and examining their properties with respect to the integral equations. Using functional analytic methods and the concept of admissibility, we obtain certain general results concerning the behavior of solutions of integral equations from which a number of results may be deduced as particular cases regarding existence, uniqueness, boundedness, and asymptotic behavior. Finally, we indicate some results on a class of general integrodifferential inequalities.

5.1. Integral inequalities

We have considered in Sect. 1.10 those integral inequalities that are reducible to differential inequalities. We shall now discuss general integral inequalities. A principle result in integral inequalities is the following.

THEOREM 5.1.1. Assume that

(i) $K \in C[J \times J \times R, R]$, $K(t, s, x)$ is monotone nondecreasing in x for each fixed (t, s), and one of the inequalities

$$\begin{aligned} x(t) &\leqslant h(t) + \int_{t_0}^{t} K(t, s, x(s))\, ds, \\ y(t) &\geqslant h(t) + \int_{t_0}^{t} K(t, s, y(s))\, ds \end{aligned} \tag{5.1.1}$$

is strict, where $x, y, h \in C[J, R]$;

(ii) $x(t_0) < y(t_0)$.

Then, we have

$$x(t) < y(t), \qquad t \geqslant t_0 . \tag{5.1.2}$$

Proof. Assume that the conclusion (5.1.2) is false.

Then, there exists a t_1 such that

$$\begin{aligned} &x(t_1) = y(t_1), \\ &x(t) < y(t), \qquad t_0 \leqslant t < t_1 . \end{aligned} \tag{5.1.3}$$

Clearly, by (ii), $t_1 > t_0$. Since K is monotone nondecreasing in x, it follows from (5.1.3) that

$$K(t_1 , s, x(s)) \leqslant K(t_1 , s, y(s)),$$

and consequently, using (5.1.1), we arrive at the inequality

$$\begin{aligned} x(t_1) &\leqslant h(t_1) + \int_{t_0}^{t_1} K(t_1 , s, x(s))\, ds \\ &\leqslant h(t_1) + \int_{t_0}^{t_1} K(t_1 , s, y(s))\, ds \\ &< y(t_1). \end{aligned}$$

This is a contradiction to the fact that $x(t_1) = y(t_1)$. Hence, the inequality (5.1.2) is true.

Let us now consider the integral operator defined by

$$K\phi = \int_{t_0}^{t} K(t, s, \phi(s))\, ds. \tag{5.1.4}$$

DEFINITION 5.1.1. We shall say that the integral operator K is monotone nondecreasing if, for any $\phi_1 , \phi_2 \in C[J, R]$ such that, for any $t_1 > t_0$,

$$\phi_1(t) < \phi_2(t), \qquad t_0 \leqslant t < t_1 ,$$

implies

$$K\phi_1(t_1) \leqslant K\phi_2(t_1).$$

THEOREM 5.1.2. Let the integral operator K defined by (5.1.4) be monotone nondecreasing. Suppose further that, for $t > t_0$,

$$x - Kx < y - Ky, \tag{5.1.5}$$

where $x, y \in C[J, R]$. Then $x(t_0) < y(t_0)$ implies

$$x(t) < y(t), \qquad t \geqslant t_0 .$$

Proof. We proceed as in Theorem 5.1.1 and obtain the relations (5.1.3). Since the integral operator K is assumed monotone nondecreasing, we have, by (5.1.3),

$$Kx(t_1) \leqslant Ky(t_1). \tag{5.1.6}$$

As a result, (5.1.5) and (5.1.6) yield

$$\begin{aligned} x(t_1) &= x(t_1) - Kx(t_1) + Kx(t_1) \\ &< y(t_1) - Ky(t_1) + Kx(t_1) \\ &\leqslant y(t_1). \end{aligned}$$

This contradicts the fact that, at $t = t_1$, $x(t_1) = y(t_1)$, and hence the proof is complete.

DEFINITION 5.1.2. A function $u \in C[J, R]$ is said to be an *under function* of the integral equation

$$x = h + Kx \tag{5.1.7}$$

if it satisfies the inequality

$$u < h + Ku.$$

Similarly, u is said to be an *over function* of (5.1.7) if it verifies the inequality

$$u > h + Ku,$$

whereas if u satisfies Eq. (5.1.7), it is said to be a *solution* of (5.1.7).

The following theorem, whose proof is a consequence of Theorem 5.1.2, shows the relation between solutions, under and over functions of the integral equation (5.1.7).

THEOREM 5.1.3. Let the integral operator K defined by (5.1.4) be monotone nondecreasing. Suppose that $x, y, z \in C[J, R]$ be an under function, a solution, and an over function of (5.1.7), respectively on $[t_0, \infty)$. Then

$$x(t_0) < y(t_0) < z(t_0)$$

implies

$$x(t) < y(t) < z(t), \qquad t \geqslant t_0 .$$

The foregoing results can easily be extended to systems of integral inequalities. We prove a result parallel to Theorem 5.1.1 only.

THEOREM 5.1.4. Assume that

(i) $K \in C[J \times J \times R^n, R^n]$, $K(t, s, x)$ is monotone nondecreasing in x for each (t, s), and one of the inequalities

$$x(t) \leqslant h(t) + \int_{t_0}^{t} K(t, s, x(s))\, ds,$$

$$y(t) \geqslant h(t) + \int_{t_0}^{t} K(t, s, y(s))\, ds$$

is strict, where $x, y, h \in C[J, R^n]$;

(ii) $x(t_0) < y(t_0)$.

These conditions imply

$$x(t) < y(t), \qquad t \geqslant t_0 .$$

Proof. If the assertion of the theorem is not true, then the set

$$Z = \bigcup_{i=1}^{n} [t \in [t_0, \infty): x_i(t) \geqslant y_i(t)]$$

is nonempty. Let $t_1 = \inf Z$. By (ii), it is clear that $t_1 > t_0$. Furthermore, since Z is closed, $t_1 \in Z$, and consequently there exists an index j such that

$$\begin{aligned} &x_j(t_1) = y_j(t_1), \\ &x_j(t) < y_j(t), \qquad t_0 \leqslant t < t_1 , \\ &x_i(t) \leqslant y_i(t), \qquad t_0 \leqslant t \leqslant t_1 , \qquad i \neq j. \end{aligned}$$

From the monotonicity of $K(t, s, x)$ in x, it results that

$$K_j(t_1, s, x(s)) \leqslant K_j(t_1, s, y(s)).$$

Hence,

$$\begin{aligned} x_j(t_1) &\leqslant h_j(t_1) + \int_{t_0}^{t_1} K_j(t_1, s, x(s))\, ds \\ &\leqslant h_j(t_1) + \int_{t_0}^{t_1} K_j(t_1, s, y(s))\, ds \\ &< y_j(t_1), \end{aligned}$$

which is an absurdity, since $x_j(t_1) = y_j(t_1)$. This shows that the set Z is empty, and the theorem is proved.

5.2. Local and global existence

Let us consider the integral equation

$$x(t) = x_0(t) + \int_{t_0}^{t} K(t, s, x(s))\, ds. \tag{5.2.1}$$

We shall, first of all, prove a local existence theorem analogous to Caratheodory's existence theorem for ordinary differential equations.

THEOREM 5.2.1. Assume that $x_0 \in C[[t_0, t_0 + a), R^n]$,

$$K \in C[[t_0, t_0 + a) \times [t_0, t_0 + a) \times R^n, R^n],$$

and the function

$$M(t) = \sup_{\substack{\|x\| \leqslant 2\beta \\ t_0 \leqslant s \leqslant t_0 + a}} \| K(t, s, x)\|$$

is summable on $[t_0, t_0 + a)$, where

$$\beta = \sup_{t_0 \leqslant t \leqslant t_0 + a} \| x_0(t)\| \quad \text{and} \quad t_0 \geqslant 0.$$

Then, there exists a number $0 < \alpha \leqslant a$ such that the integral equation (5.2.1) has at least one solution on $[t_0, t_0 + \alpha]$.

Proof. Since the proof is similar to the existence theorem of Caratheodory, we shall be brief. Consider the sequence of approximations $\{\phi_j\}$ defined by

$$\phi_j(t) = x_0(t), \qquad t_0 \leqslant t \leqslant t_0 + (\alpha/j) \qquad (j = 1, 2,...),$$

$$\phi_j(t) = x_0(t) = \int_{t_0}^{t_0 - (\alpha/j)} K(t - (\alpha/j), s, \phi_j(s))\, ds, \qquad t_0 + (\alpha/j) < t \leqslant t_0 + \alpha$$
$$(j = 1, 2,...).$$

It is easy to show that the sequence $\{\phi_j\}$ forms a family of uniformly bounded and equicontinuous functions on $[t_0, t_0 + \alpha]$, where α is defined by the relation $P(\alpha) \leqslant \beta$, where

$$P(t) = \int_{t_0}^{t} M(\xi)\, d\xi. \tag{5.2.2}$$

Since $P(0) = 0$, $P(t)$ is continuous and monotonic nondecreasing, the existence of such an α is clear. Then $\{\phi_j\}$ contains a subsequence con-

verging uniformly on $[t_0, t_0 + \alpha)$ to a limit function $x(t)$, which can be shown to satisfy (5.2.1), using the usual techniques. The proof is complete.

A global existence theorem that includes Theorem 2.1.2 can be proved using Tychonoff's Theorem 2.1.1.

THEOREM 5.2.2. Let $K \in C[J \times J \times R^n, R^n]$, $G \in C[J \times J \times R_+, R_+]$, $G(t, s, u)$ be monotone nondecreasing in u for each (t, s), and

$$\| K(t, s, x)\| \leqslant G(t, s, \| x \|). \tag{5.2.3}$$

Assume that, for every continuous function $u_0(t) > 0$,

$$u(t) = u_0(t) + \int_{t_0}^{t} G(t, s, u(s))\, ds \tag{5.2.4}$$

possesses a solution $u(t)$ existing on $[t_0, \infty)$. Then, for any $x_0 \in C[J, R^n]$ such that $\| x_0(t)\| \leqslant u_0(t)$, there exists a solution $x(t)$ of the integral equation (5.2.1.) on $[t_0, \infty)$ satisfying

$$\| x(t)\| \leqslant u(t), \qquad t \geqslant t_0 .$$

Proof. The proof is very much the same as that of Theorem 2.1.2. In the present case, the integral operator T defined by (2.1.5) takes the form

$$T(x)(t) = x_0(t) + \int_{t_0}^{t} K(t, s, x(s))\, ds.$$

The space of continuous functions B, the topology on B, and the closed, convex, and bounded set B_0 remain the same as in the proof of Theorem 2.1.2. The operator T is compact in the topology of B, and hence $\overline{T(B_0)}$ is compact, since the set B_0 is bounded. To show $T(B_0) \subset B_0$, we notice that

$$\begin{aligned} \| T(x)(t)\| &\leqslant \| x_0(t)\| + \int_{t_0}^{t} \| K(t, s, x(s))\|\, ds \\ &\leqslant u_0(t) + \int_{t_0}^{t} G(t, s, \| x(s)\|)\, ds \\ &\leqslant u_0(t) + \int_{t_0}^{t} G(t, s, u(s))\, ds = u(t), \end{aligned}$$

using the monotony of G. We can conclude the validity of the theorem, on the basis of Theorem 2.1.2.

The notion of maximal and minimal solutions may be introduced.

DEFINITION 5.2.1. Let $r(t)$ be a solution of the integral equation (5.2.1) existing on $[t_0, t_0 + a)$. Then $r(t)$ is said to be the maximal solution of (5.2.1) if, for every solution $x(t)$ of (5.2.1) existing on $[t_0, t_0 + a)$, the inequality

$$x(t) \leqslant r(t), \qquad t \in [t_0, t_0 + a)$$

is verified. By reversing the preceding inequality, we may define the minimal solution of (5.2.1).

The existence of maximal and minimal solutions may be proved under the hypothesis of Theorem 5.2.1.

THEOREM 5.2.3. Let the hypotheses of Theorem 5.2.1 be satisfied. Suppose that $K(t, s, x)$ is monotone nondecreasing in x for each (t, s). Then there exists a maximal solution and a minimal solution on $[t_0, t_0 + \alpha]$ for a certain $\alpha > 0$.

Proof. We shall indicate the proof of the existence of maximal solution only. Consider, for some arbitrarily small vector $\epsilon_0 > 0$, the integral equation

$$x(t) = x_0(t) + \epsilon_0 + \int_{t_0}^{t} K(t, s, x(s))\, ds.$$

On the basis of Theorem 5.2.1, there exists an $\alpha > 0$ such that there is a solution $x(t, \epsilon_0)$ on $[t_0, t_0 + \alpha]$. Let $0 < \epsilon_2 < \epsilon_1 \leqslant \epsilon_0$. Then, we have

$$\begin{aligned} x(t_0, \epsilon_2) &< x(t_0, \epsilon_1), \\ x(t, \epsilon_2) &\leqslant x_0(t) + \epsilon_2 + \int_{t_0}^{t} K(t, s, x(s, \epsilon_2))\, ds, \\ x(t, \epsilon_1) &> x_0(t) + \epsilon_2 + \int_{t_0}^{t} K(t, s, x(s, \epsilon_1))\, ds. \end{aligned}$$

An application of Theorem 5.1.1 yields

$$x(t, \epsilon_2) < x(t, \epsilon_1), \qquad t \in [t_0, t_0 + \alpha].$$

Since the family of functions $\{x(t, \epsilon)\}$ are equicontinuous and uniformly bounded on $[t_0, t_0 + \alpha]$, it follows by Theorem 1.1.1 that there exists a decreasing sequence $\{\epsilon_n\}$ tending to zero as $n \to \infty$, and the uniform limit

$$r(t) = \lim_{n \to \infty} x(t, \epsilon_n)$$

exists on $[t_0, t_0 + \alpha]$. It can be easily shown that $r(t)$ is a solution of (5.2.1). Furthermore, to show that $r(t)$ is the desired maximal solution

of (5.2.1) on $[t_0, t_0 + \alpha]$, let $x(t)$ be any solution of (5.2.1) defined on $[t_0, t_0 + \alpha]$. Then, on the strength of Theorem 5.1.1, it follows that, for $\epsilon \leqslant \epsilon_0$,

$$x(t) < x(t, \epsilon), \qquad t \in [t_0, t_0 + \alpha].$$

The uniqueness of the maximal solution shows that $x(t, \epsilon)$ tends uniformly to $r(t)$ on $[t_0, t_0 + \alpha]$, and therefore the proof is complete.

5.3. Comparison theorems

As in ordinary differential equations, an important technique is concerned with comparing a function satisfying an integral inequality by the maximal solution of the corresponding integral equation. The following theorem is a result of this type.

THEOREM 5.3.1. Let $G \in C[J \times J \times R_+, R]$, $G(t, s, u)$ be monotone nondecreasing in u for each (t, s), and

$$m(t) \leqslant m_0(t) + \int_{t_0}^{t} G(t, s, m(s))\, ds, \qquad t \geqslant t_0,$$

where $m \in C[J, R_+]$. Suppose that $r(t)$ is the maximal solution of the scalar integral equation

$$u(t) = u_0(t) + \int_{t_0}^{t} G(t, s, u(s))\, ds \tag{5.3.1}$$

existing on J. Then, the inequality $m(t_0) \leqslant u_0(t_0)$ implies

$$m(t) \leqslant r(t), \qquad t \geqslant t_0. \tag{5.3.2}$$

Proof. Let $u(t, \epsilon)$ be any solution of the integral equation

$$u(t) = u_0(t) + \epsilon + \int_{t_0}^{t} G(t, s, u(s))\, ds$$

for $\epsilon > 0$ sufficiently small. Since $\lim_{\epsilon \to 0} u(t, \epsilon) \equiv r(t)$, it is enough to show that

$$m(t) < u(t, \epsilon), \qquad t \geqslant t_0. \tag{5.3.3}$$

Observe that $m(t_0) < u(t_0, \epsilon)$ and

$$u(t, \epsilon) > u_0(t) + \int_{t_0}^{t} G(t, s, u(s, \epsilon))\, ds.$$

Hence, an application of Theorem 5.1.1 shows that the inequality (5.3.3) is valid. This establishes the theorem.

We shall next prove an extension of the result of Theorem 5.3.1 to systems of integral inequalities. The proof that will be presented is simple and short and makes use of the partial ordering in R^n.

Let us introduce the relation $\leqslant$ in R^n, namely, we set, for any two elements $x, y \in R^n$,

$$x \leqslant y \quad \text{iff} \quad x_i \leqslant y_i \quad \text{for each} \quad i = 1, 2, \ldots, n. \tag{5.3.4}$$

This relation induces a partial ordering in R^n, and it is easy to see that, for any bounded set $A \subset R^n$, there exists the sup A with respect to the relation (5.3.4), which is

$$\sup A = \min[z \in R^n : x \leqslant z \quad \text{for each} \quad x \in A]. \tag{5.3.5}$$

In fact, we need (5.3.5) only for two elements sets, in which case we have

$$\begin{gathered} \sup[x, y] = z = (z_1, z_2, \ldots, z_n), \\ \text{where} \quad z_i = \max(x_i, y_i), \end{gathered} \tag{5.3.6}$$

x_i, y_i being the components of x and y, respectively. We are now in a position to prove the following:

THEOREM 5.3.2. Let $K \in C[J \times J \times R^n, R^n]$, $K(t, s, x)$ be monotonic nondecreasing in x for each (t, s), and

$$x(t) \leqslant x_0(t) + \int_{t_0}^{t} K(t, s, x(s))\, ds, \tag{5.3.7}$$

where $x, x_0 \in C[J, R^n]$. Assume that $r(t)$ is the maximal solution of

$$u(t) = x_0(t) + \int_{t_0}^{t} K(t, s, u(s))\, ds \tag{5.3.8}$$

existing on $[t_0, \infty)$. Then

$$x(t) \leqslant r(t), \qquad t \geqslant t_0. \tag{5.3.9}$$

Proof. Define

$$F(t, s, y) = K(t, s, \sup[y, x(t)]). \tag{5.3.10}$$

By (5.3.6), $x(t) \leqslant \sup[y, x(t)]$, and therefore it follows, from the monotonicity of K and (5.3.10), that

$$F(t, s, y) \geqslant K(t, s, x(t)) \qquad \text{for each} \quad y. \tag{5.3.11}$$

Let $r^*(t)$ be the maximal solution of

$$u(t) = x_0(t) + \int_{t_0}^{t} F(t, s, u(s))\, ds$$

existing on $[t_0, \infty)$. Then, using (5.3.11) and (5.3.7), we get

$$\begin{aligned} r^*(t) &= x_0(t) + \int_{t_0}^{t} F(t, s, r^*(s))\, ds \\ &\geqslant x_0(t) + \int_{t_0}^{t} K(t, s, x(s))\, ds \\ &\geqslant x(t). \end{aligned} \tag{5.3.12}$$

It then results from (5.3.12) and (5.3.6) that

$$\sup[r^*(t), x(t)] = r^*(t),$$

and consequently, by (5.3.10),

$$F(t, s, r^*(t)) = K(t, s, r^*(t)).$$

Thus, $r^*(t)$ is also the maximal solution of (5.3.8). Hence, (5.3.12) proves the desired result (5.3.9). The proof is complete.

COROLLARY 5.3.1. Let $f \in C[J \times R^n, R^n]$, $f(t, x)$ be monotonic nondecreasing in x for each t and

$$x(t) \leqslant x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

where $x \in C[J, R^n]$. Suppose that $r(t)$ is the maximal solution of

$$y' = f(t, y), \qquad y(t_0) = x_0,$$

existing on $[t_0, \infty)$. Then,

$$x(t) \leqslant r(t), \qquad t \geqslant t_0.$$

5.4. Approximate solutions, bounds, and uniqueness

Let us define an approximate solution of the integral equation (5.2.1).

DEFINITION 5.4.1. Let $x \in C[J, R^n]$, and satisfy

$$\left\| x(t) - x_0(t) - \int_{t_0}^{t} K(t, s, x(s))\, ds \right\| \leqslant \delta(t), \tag{5.4.1}$$

where $\delta \in C[J, R_+]$. Then $x(t)$ is said to be a δ-approximate solution of (5.2.1).

The difference between an approximate solution and a solution is given by the following result.

THEOREM 5.4.1. Assume that

(i) $K \in C[J \times J \times R^n, R^n]$, $G \in C[J \times J \times R_+, R_+]$, $G(t, s, u)$ is monotonic nondecreasing in u for each (t, s), and

$$\| K(t, s, x) - K(t, s, y)\| \leqslant G(t, s, \| x - y \|); \tag{5.4.2}$$

(ii) $x(t, \delta)$ is a δ-approximate solution of (5.2.1), where $\delta \in C[J, R_+]$;

(iii) $r(t)$ is the maximal solution of

$$u(t) = \delta(t) + \int_{t_0}^{t} G(t, s, u(s))\, ds \tag{5.4.3}$$

existing on $[t_0, \infty)$.

Then, if $y(t)$ is any solution of (5.2.1) existing on $[t_0, \infty)$, we have

$$\| x(t, \delta) - y(t)\| \leqslant r(t), \qquad t \geqslant t_0 . \tag{5.4.4}$$

Proof. Consider the function

$$m(t) = \| x(t, \delta) - y(t)\|,$$

where $x(t, \delta)$ and $y(t)$ are δ-approximate solution and solution of (5.2.1), respectively. Then, using (5.4.1) and (5.4.2), we get

$$\begin{aligned} m(t) &= \| x(t, \delta) - x_0(t) - \int_{t_0}^{t} K(t, s, x(s, \delta))\, ds \| \\ &\quad + \int_{t_0}^{t} \| K(t, s, x(s, \delta)) - K(t, s, y(s))\|\, ds \\ &\leqslant \delta(t) + \int_{t_0}^{t} G(t, s, m(s))\, ds. \end{aligned}$$

An application of Theorem 5.3.1 now yields

$$m(t) = \| x(t, \delta) - y(t)\| \leqslant r(t), \qquad t \geqslant t_0 ,$$

and the proof is complete.

The next theorem offers an estimate of the growth of solutions of (5.2.1).

THEOREM 5.4.2. Suppose that

(i) $K \in C[J \times J \times R^n, R^n]$, $G \in C[J \times J \times R_+, R_+]$, $G(t, s, u)$ is monotonic nondecreasing in u for each (t, s), and

$$\| K(t, s, x)\| \leqslant G(t, s, \| x \|); \tag{5.4.5}$$

(ii) $r(t)$ is the maximal solution of (5.3.1) existing on $[t_0, \infty)$;

(iii) $x(t)$ is any solution of (5.2.1) existing on $[t_0, \infty)$ such that $\| x_0(t)\| \leqslant u_0(t)$.

Then, we have

$$\| x(t)\| \leqslant r(t), \qquad t \geqslant t_0 . \tag{5.4.6}$$

Proof. If $m(t) = \| x(t)\|$, we have, by (5.4.5), the integral inequality

$$\begin{aligned} m(t) &\leqslant \| x_0(t)\| + \int_{t_0}^{t} \| K(t, s, x(s))\| \, ds \\ &\leqslant u_0(t) + \int_{t_0}^{t} G(t, s, m(s)) \, ds, \end{aligned}$$

and, consequently, Theorem 5.3.1 assures (5.4.6).

THEOREM 5.4.3. Assume that

(i) $K_1, K_2 \in C[J \times J \times R^n, R^n]$, $G \in C[J \times J \times R_+, R_+]$, $G(t, s, u)$ is monotonic nondecreasing in u for each (t, s), and

$$\| K_1(t, s, x) - K_2(t, s, y)\| \leqslant G(t, s, \| x - y \|); \tag{5.4.7}$$

(ii) $x_0, y_0 \in C[J, R^n]$, and $x(t)$, $y(t)$ are any two solutions of

$$\begin{aligned} x(t) &= x_0(t) + \int_{t_0}^{t} K_1(t, s, x(s)) \, ds, \\ y(t) &= y_0(t) + \int_{t_0}^{t} K_2(t, s, y(s)) \, ds, \end{aligned}$$

respectively;

(iii) $r(t)$ is the maximal solution of (5.3.1) such that

$$\| x_0(t) - y_0(t)\| \leqslant u_0(t), \qquad t \geqslant t_0 .$$

Under these assumptions, we have

$$\| x(t) - y(t)\| \leqslant r(t), \qquad t \geqslant t_0 .$$

Proof. The proof is an easy modification of the proof of Theorem 5.4.2. For, setting $m(t) = \| x(t) - y(t)\|$ and using (5.4.7), we obtain

$$\begin{aligned} m(t) &\leqslant \| x_0(t) - y_0(t)\| + \int_{t_0}^{t} \| K_1(t, s, x(s)) - K_2(t, s, y(s))\| \, ds \\ &\leqslant u_0(t) + \int_{t_0}^{t} G(t, s, m(s)) \, ds. \end{aligned}$$

The desired result follows from Theorem 5.3.1.

A uniqueness theorem of Perron type may now be stated.

THEOREM 5.4.4. Suppose that

(i) $G \in C[[t_0, t_0 + a] \times [t_0, t_0 + a] \times R_+, R_+]$, $G(t, s, 0) \equiv 0$, $G(t, s, u)$ is monotone nondecreasing in u for each (t, s), and $u(t) \equiv 0$ is the only solution of the integral equation

$$u(t) = \int_{t_0}^{t} G(t, s, u(s)) \, ds \tag{5.4.8}$$

on $[t_0, t_0 + a]$;

(ii) $K \in C[[t_0, t_0 + a] \times [t_0, t_0 + a] \times R^n, R^n]$, and

$$\| K(t, s, x) - K(t, s, y)\| \leqslant G(t, s, \| x - y \|).$$

Then, there exists at most one solution of (5.2.1) on $t_0 \leqslant t \leqslant t_0 + a$.

Proof. Let $x(t)$, $y(t)$ be two solutions of (5.2.1) existing on $[t_0, t_0 + a]$. Setting $m(t) = \| x(t) - y(t)\|$ and arguing as before, we get

$$m(t) \leqslant \int_{t_0}^{t} G(t, s, m(s)) \, ds,$$

which implies, in view of Theorem 5.3.1, that

$$m(t) \leqslant r(t), \qquad t \geqslant t_0,$$

where $r(t)$ is the maximal solution of (5.4.8). Since $m(t_0) = 0$ and $u(t) \equiv 0$ is the only solution of (5.4.8), the assertion of the theorem is immediate.

5.5. Asymptotic behavior

In this section, we shall investigate the asymptotic behavior of solutions of a Volterra integral equation of the form

$$x'(t) = - \int_{0}^{t} a(t - s) \, g(x(s)) \, ds. \tag{5.5.1}$$

This is equivalent to considering the integral equation

$$x(t) = x(0) - \int_0^t A(t - s)\, g(x(s))\, ds,$$

where

$$A(t) = \int_0^t a(\tau)\, d\tau.$$

Before proceeding further, we shall prove some elementary lemmas.

LEMMA 5.5.1. *Let* $f \in C[J, R_+]$, $f''(t)$ *exist on* $0 < t < \infty$, *and* $f'(t) \leqslant 0$, $f''(t) \geqslant -k > -\infty$ *for* $0 < t < \infty$, k *being some positive constant. Then* $f'(t) \to 0$ *as* $t \to \infty$.

Proof. Suppose that the conclusion is false. Then, by hypothesis, there exists a $\lambda > 0$ and a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that

$$f'(t_n) \leqslant -\lambda < 0.$$

Consider the intervals

$$I_n = [t_n - (\lambda/2k), t_n] \qquad \text{for} \quad n \geqslant N,$$

where $t_n - (\lambda/2k) > 0$ for $n \geqslant N$. Using mean value theorem, we obtain

$$\begin{aligned} f'(t) &= f'(t_n) + f''(\theta)(t - t_n) \\ &\leqslant -\lambda + \tfrac{1}{2}\lambda = -\tfrac{1}{2}\lambda, \end{aligned}$$

where $t \in I_n$, $n \geqslant N$, and $t < \theta = \theta(t, t_n) < t_n$. Applying mean value theorem again, it follows that

$$f\left(t_n - \frac{\lambda}{2k}\right) - f(t_n) \geqslant \frac{\lambda}{2}\left(\frac{\lambda}{2k}\right) = \frac{\lambda^2}{4k}, \qquad n \geqslant N,$$

which clearly contradicts the fact that $f(t)$ decreases to $f(\infty) \geqslant 0$ as $t \to \infty$, and thus completes the proof.

REMARK 5.5.1. If $f''(t)$ is bounded from above rather than from below, Lemma 5.5.1 remains valid. Also, instead of $f''(t) \geqslant -k$, we may ask the right second derivative only.

LEMMA 5.5.2. Assume that $a \in C[J, R]$, $(-1)^i a^i(t) \geqslant 0$ for $0 < t < \infty$ $(i = 0, 1, 2, 3)$, and $a(t) \not\equiv a(0)$. Then, either $-a'(t)$, $a''(t) > 0$ for $0 < t < \infty$, or there exists a $t_0 > 0$ such that $-a'(t)$, $a''(t) > 0$ for $0 < t < t_0$ and $a(t) \equiv a(t_0) = a(\infty) \geqslant 0$ for $t_0 \leqslant t < \infty$.

Proof. If there exists a $t_0 \geqslant 0$ such that $a'(t_0) = 0$, then, since $-a'(t) \geqslant 0$, $a''(t) \geqslant 0$, it follows that $a(t) = a(t_0) \geqslant 0$ for $t_0 \leqslant t < \infty$. This implies $t_0 > 0$ in view of the fact $a(t) \not\equiv a(0)$. Hence, there exists a $t_1 > 0$ such that $a'(t_1) < 0$, and thus $a'(t) \leqslant a'(t_1) < 0$ for $0 < t \leqslant t_1$. Suppose that $a''(t_1) = 0$. Then, as $-a'''(t)$, $a''(t) \geqslant 0$, this means that $a''(t) \equiv 0$ for $t_1 \leqslant t < \infty$. Hence,

$$a(t) = a(t_1) + a'(t_1)(t - t_1), \qquad t_1 \leqslant t < \infty,$$

which contradicts $a(t) \geqslant 0$ for t sufficiently large. Consequently, $a''(t_1) > 0$, and thus $a''(t) \geqslant a''(t_1) > 0$ for $0 < t \leqslant t_1$. The conclusion of the lemma is immediate.

REMARK 5.5.2. Although $a(\infty) \geqslant 0$ is not necessarily zero, as a result of Lemma 5.5.1, we derive that $a'(t)$ increases to zero and $a''(t)$ decreases to zero as $t \to \infty$.

LEMMA 5.5.3. If $a \in C[J, R]$, $(-1)^i a^i(t) \geqslant 0$ for $0 < t < \infty$, then

$$ta'(t) \to 0, \qquad t^2 a''(t) \to 0 \qquad \text{as} \quad t \to 0^+, \tag{5.5.2}$$

and

$$a'(t), \qquad ta''(t), \qquad t^2 a'''(t) \in L_1(0, \infty). \tag{5.5.3}$$

Proof. By the mean value theorem and the monotonicity of $a'(t)$, we deduce that

$$a(t) - a(0) = ta'(\xi) \leqslant ta'(t) \leqslant 0, \qquad 0 < \xi < t < \infty,$$

from which $\lim_{t \to 0^+} ta'(t) = 0$ follows.

By the second differences and the monotonicity of $a''(t)$, we have

$$\begin{aligned} a(t) - 2a\left(\frac{t}{2}\right) + a(0) &= \left(\frac{t}{2}\right)^2 a''(\xi) \\ &\geqslant \left(\frac{t}{2}\right)^2 a''(t) \geqslant 0, \qquad 0 < \xi < t < \infty, \end{aligned}$$

which yields $\lim_{t \to 0^+} t^2 a''(t) = 0$.

As the integrand is of constant sign, $a'(t) \in L_1(0, \infty)$ follows from

$$\begin{aligned} \int_0^\infty a'(s)\, ds &= \lim_{\substack{t \to \infty \\ \epsilon \to 0^+}} \int_\epsilon^t a'(s)\, ds = \lim_{\substack{t \to \infty \\ \epsilon \to 0^+}} [a(t) - a(\epsilon)] \\ &= a(\infty) - a(0). \end{aligned}$$

By a similar reasoning, we can show the other two statements in (5.5.3). Hence, the proof is complete.

LEMMA 5.5.4. Let $b(t)$ be defined on $0 < t \leqslant T$, $b'(t)$ exist and be finite on $0 < t \leqslant T$, and $b'(t) \in L_1(\epsilon, T)$ for each $0 < \epsilon < T$. Let $q(t, s)$, $\partial q(t, s)/\partial t$ be continuous on $0 < s$, $t \leqslant T$ in t, s. Suppose that

$$b(\epsilon)\, q(t + \epsilon, t) \to \gamma(t) \qquad \text{as} \quad \epsilon \to 0^+ \quad \text{on} \quad 0 < t < T,$$

where $\gamma \in C[[0, T], R]$. Assume that there exists a $\phi(\xi) \in L_1(0, T)$ such that

$$| b(\xi)\, q(t, t - \xi)|, | b'(\xi)\, q(t, t - \xi)|, \left| b(\xi) \frac{\partial q(t, t - \xi)}{\partial t} \right| \leqslant \phi(\xi)$$

for $0 < \xi \leqslant t \leqslant T$. Then,

$$f(t) = \int_0^t b(t - s)\, q(t, s)\, ds$$

is continuously differentiable, and $f'(t) = h(t)$, where

$$h(t) = \gamma(t) + \int_0^t b'(t - s)\, q(t, s)\, ds + \int_0^t b(t - s) \frac{\partial q(t, s)}{\partial t}\, ds \qquad (5.5.4)$$

for $0 \leqslant t \leqslant T$.

Proof. Define $h(t)$ by (5.5.4). It follows readily from the hypothesis that $h \in C[[0, T], R]$. Also,

$$\int_0^t h(s)\, ds = \int_0^t \gamma(s)\, ds + \int_0^t \left[\int_\tau^t b'(s - \tau)\, q(s, \tau)\, ds\right] d\tau$$
$$+ \int_0^t \left[\int_0^s b(s - \tau) \frac{\partial q(s, \tau)}{\partial s}\, d\tau\right] ds, \qquad (5.5.5)$$

where the interchange of order of integration is easily justified by the hypothesis and Fubini's theorem. We note that the assumption of the lemma implies that $b(t)$ is absolutely continuous on $0 < \epsilon \leqslant t \leqslant T$, and this yields the second equality in

$$\int_\tau^t b'(s - \tau)\, q(s, \tau)\, ds = \lim_{\epsilon \to 0^+} \int_{\tau+\epsilon}^t b'(s - \tau)\, q(s, \tau)\, ds$$
$$= \lim_{\epsilon \to 0^+} \left[b(s - \tau)\, q(s, \tau) \Big|_{\tau+\epsilon}^t - \int_{\tau+\epsilon}^t b(s - \tau) \frac{\partial q(s, \tau)}{\partial s}\, ds \right]$$
$$= b(t - \tau)\, q(t, \tau) - \gamma(\tau) - \int_\tau^t b(s - \tau) \frac{\partial q(s, \tau)}{\partial s}\, ds,$$
$$0 < \tau < t. \qquad (5.5.6)$$

Combining the relations (5.5.5) and (5.5.6), we obtain

$$f(t) = \int_0^t h(s)\, ds$$

after an interchange of the order of integration. The conclusion of the lemma now follows readily.

We now prove

THEOREM 5.5.1. Assume that

(i) $a \in C[J, R]$, $(-1)^i a^i(t) \geqslant 0$, $0 < t < \infty$, $i = 0, 1, 2, 3$, and $a(t) \not\equiv a(0)$;

(ii) $g \in C[R, R]$, $xg(x) > 0$, $x \neq 0$, and

$$G(x) = \int_0^x g(\xi)\, d\xi \to \infty \qquad \text{as} \quad |\, x \,| \to \infty;$$

(iii) $u(t)$ is any solution of (5.5.1) existing on $0 \leqslant t < \infty$.

Under these assumptions,

$$\lim_{t\to\infty} u^i(t) = 0 \qquad (i = 0, 1, 2). \tag{5.5.7}$$

Proof. Differentiating (5.5.1), we get

$$x''(t) + a(0)\, g(x(t)) = -\int_0^t a'(t-s)\, g(x(s))\, ds. \tag{5.5.8}$$

Whenever we refer to (5.5.1) and (5.5.8), we mean the identities that result from substituting $u(t)$ into them. The possibility of none of $a'(0)$, $a''(0)$, $a'''(0)$ being finite necessitates that a little care be exercised in handling certain integrals that arise. In all the cases, the arguments used in the preceding lemmas supply the rigor, and hence, in this proof, we tacitly assume such considerations whenever they are relevant.

Consider the function

$$\begin{aligned} V(t) &= G(u(t)) + \tfrac{1}{2}a(t)\left[\int_0^t g(u(s))\, ds\right]^2 \\ &\quad - \tfrac{1}{2}\int_0^t a'(t-s)\left[\int_s^t g(u(\tau))\, d\tau\right]^2 ds \geqslant 0. \end{aligned} \tag{5.5.9}$$

Using (5.5.1) and integrating by parts, we obtain

$$\begin{aligned} V'(t) &= \tfrac{1}{2}a'(t)\left[\int_0^t g(u(s))\, ds\right]^2 \\ &\quad - \tfrac{1}{2}\int_0^t a''(t-s)\left[\int_s^t g(u(\tau))\, d\tau\right]^2 ds \leqslant 0, \end{aligned} \tag{5.5.10}$$

which implies that

$$G(u(t)) \leqslant V(t) \leqslant V(0) = G(u_0),$$

where $u_0 = u(0)$. It then follows from assumption (ii) that

$$| u(t)| \leqslant \beta, \qquad t \in J, \tag{5.5.11}$$

where $\beta = \beta(u_0) \to 0$ as $u_0 \to 0$. In succeeding formulas, β will not necessarily be the same as in (5.5.11). However, it will have the same property.

From (5.5.3), (5.5.8), and (5.5.11), we derive that

$$| u''(t)| \leqslant \beta, \qquad t \in J. \tag{5.5.12}$$

The inequalities (5.5.11), (5.5.12), and the mean value theorem show that

$$| u'(t)| \leqslant \beta, \qquad t \in J. \tag{5.5.13}$$

Integration by parts and (5.5.8) yield

$$\begin{aligned} V''(t) = {} & \tfrac{1}{2}a''(t)\left[\int_0^t g(u(s))\, ds\right]^2 \\ & - \tfrac{1}{2}\int_0^t a'''(t-s)\left[\int_s^t g(u(\tau))\, d\tau\right]^2 ds \\ & - g(u(t))[u''(t) + a(0)\, g(u(t))]. \end{aligned}$$

By Lemma 5.5.3, $t^2a''(t) \in L_1(0, \infty)$. This, together with (5.5.11), (5.5.12), and Lemma 5.5.3, implies that

$$| V''(t)| \leqslant \beta, \qquad t \in J.$$

By Lemma 5.5.1, it follows that

$$V'(t) \to 0 \qquad \text{as} \quad t \to \infty.$$

Hence,

$$\lim_{t\to\infty} \int_0^t a''(t-s)\left[\int_s^t g(u(\tau))\, d\tau\right]^2 ds = 0,$$

which assures, in view of $-a'''(t),\ a''(t) \geqslant 0$,

$$\lim_{t\to\infty} a''(T)\int_{t-T}^t \left[\int_s^t g(u(\tau))\, d\tau\right]^2 ds = 0,$$

for every $0 < T < \infty$. Choose $T_0 > 0$ arbitrarily if the first alternative of Lemma 5.5.2 holds, and choose $0 < T_0 < t_0$ if the second one does. Then, clearly

$$\lim_{t\to\infty} \int_{t-T}^{t} \left[\int_{s}^{t} g(u(\tau))\, d\tau\right]^2 ds = 0, \tag{5.5.14}$$

$0 < T \leqslant T_0$. Suppose that $\lim_{t\to\infty} u(t) \neq 0$. Then, there exists a $\lambda > 0$ and a sequence $\{t_n\}$, $t_1 > 0$, $t_n \to \infty$ as $n \to \infty$, such that

$$| u(t_n)| \geqslant \lambda.$$

This, together with the relations (5.5.8), (5.5.13), and the mean value theorem, implies the existence of a $\delta > 0$ and a $\mu > 0$, where $0 < \delta \leqslant \min(T_0, t_1)$, such that

$$| g(u(\tau))| \geqslant \mu \qquad \text{for} \quad t_n - \delta \leqslant \tau \leqslant t_n .$$

As a result, we have

$$\begin{aligned} \int_{t_n-\delta}^{t_n} \left[\int_{s}^{t_n} g(u(\tau))\, d\tau\right]^2 ds &\geqslant \mu^2 \int_{t_n-\delta}^{t_n} (t_n - s)^2\, ds \\ &= \tfrac{1}{3}\mu^2\delta^2 > 0 \qquad (n = 1, 2,\ldots), \end{aligned}$$

which contradicts (5.5.14). Thus, $\lim_{t\to\infty} u(t) = 0$ is established.

Formula (5.5.7), $i = 1$, follows from (5.5.7), $i = 0$, (5.5.12), and the mean value theorem by employing an argument similar to the proof of Lemma 5.5.1. Similarly, formula (5.5.7), $i = 2$, follows from (5.5.7), $i = 0$, assumption (ii), (5.5.8), and the fact that $a'(t) \in L_1(0, \infty)$. This completes the proof of the theorem.

5.6. Perturbed integral equations

Corresponding to the integral equation (5.5.1), let us first consider the perturbed equation

$$x'(t) = -\int_0^t a(t - s)\, g(x(s))\, ds - b(t) + f(t). \tag{5.6.1}$$

As in the previous section, the letter β denotes a finite *a priori* bound that may vary from time to time. Concerning Eq. (5.6.1), we have the following result.

THEOREM 5.6.1. Assume that

(i) $a \in C[J, R]$, $(-1)^i a^{(i)}(t) \geqslant 0$ for $0 < t < \infty$, $i = 0, 1, 2$;

(ii) $g \in C[R, R]$, $xg(x) \geqslant 0$, $G(x) = \int_0^x g(\xi)\, d\xi \to \infty$ as $|x| \to \infty$, and $|g(x)| \leqslant K_1(1 + G(x))$ for some $K_1 > 0$;

(iii) $b \in C[J, R]$, $b'(t)$ exists and is continuous on $0 < t < \infty$;

(iv) there exists a $\gamma \in C[J, R]$ such that $\gamma(t)$ is continuously differentiable on $0 < t < \infty$ and

$$b^2(t) \leqslant a(t)\,\gamma(t), \qquad (b'(t))^2 \leqslant a'(t)\,\gamma'(t), \qquad 0 < t < \infty;$$

(v) $f \in C[J, R]$, and $f \in L_1(0, \infty)$.

Then, if $x(t)$ is a solution of (5.6.1) on $0 \leqslant t < \infty$, we have

$$|x(t)| \leqslant K, \qquad t \in J. \tag{5.6.2}$$

Suppose that, in addition:

(vi) $-a'''(t) \geqslant 0$, $0 < t < \infty$, $xg(x) > 0$, $x \neq 0$, $g(x)$ is differentiable on R, and $b''(t)$, $\gamma''(t)$ exist on $0 < t < \infty$;

(vii) either $(b''(t))^2 \leqslant a''(t)\,\gamma''(t)$, $0 < t < \infty$, or $|b'(t)|$, $|tb''(t)|$, $|\gamma''(t)| \leqslant K$, for some $K > 0$, on $\mu \leqslant t < \infty$, $\mu > 0$ being some number;

(viii) $f(t)$ is continuously differentiable on $\mu \leqslant t < \infty$ and bounded. Then,

$$|x''(t)| \leqslant K, \qquad \mu \leqslant t < \infty,$$

and, if also $a(t) \not\equiv a(0)$,

$$\lim_{t\to\infty} x^i(t) = 0 \qquad (i = 0, 1). \tag{5.6.3}$$

Proof. We shall first prove (5.6.2). For $t \in J$, define

$$\begin{aligned} E(t) = G(x(t)) &+ \tfrac{1}{2}a(t)\left[\int_0^t g(x(s))\, ds\right]^2 \\ &+ b(t)\int_0^t g(x(s))\, ds + \tfrac{1}{2}\gamma(t) \\ &- \tfrac{1}{2}\int_0^t a'(t-\tau)\left[\int_s^t g(x(s))\, ds\right]^2 d\tau, \end{aligned} \tag{5.6.4}$$

$$F(t) = \int_0^t |f(\tau)|\, d\tau, \tag{5.6.5}$$

and

$$V(t) = [1 + E(t)] \exp(-K_1 F(t)). \tag{5.6.6}$$

From assumptions (i), (ii), and (iv), it is evident that $E(t)$, $V(t) \geqslant 0$. Differentiation of (5.6.6) yields, after some calculation involving an integration by parts,

$$\begin{aligned} V'(t) = -K_1 \,|f(t)|\, V(t) + \Big\{ & g(x(t)) f(t) + \tfrac{1}{2} a'(t) \Big(\int_0^t g(x(s))\, ds \Big)^2 \\ & + b'(t) \int_0^t g(x(s))\, ds + \tfrac{1}{2}\gamma'(t) - \tfrac{1}{2} \int_0^t a''(t-\tau) \Big(\int_\tau^t g(x(s))\, ds \Big)^2 d\tau \Big\} \\ & \times \exp(-K_1 F(t)). \end{aligned} \tag{5.6.7}$$

Hence, by (i), (iv), and (5.6.7), we see that

$$V'(t) \leqslant \{-K_1 - K_1 G(x(t)) + |g(x(t))|\} \,|f(t)|\, \exp(-K_1 F(t)),$$

which, together with assumption (ii), implies that

$$V'(t) \leqslant 0.$$

Therefore, it follows that

$$\begin{aligned} G(x(t)) \exp[-K_1 F(t)] &\leqslant V(t) \leqslant V(0) \\ &= 1 + G(x(0)) + \tfrac{1}{2}\gamma(0), \end{aligned}$$

and so

$$G(x(t)) \leqslant [1 + G(x(0)) + \tfrac{1}{2}\gamma(0)] \exp\Big(K_1 \int_0^\infty |f(t)|\, dt \Big).$$

From this inequality, the truth of (5.6.2) is clear in view of the assumptions on f and G.

To prove the second part, we differentiate (5.6.1) to obtain

$$x''(t) = -a(0)\, g(x(t)) - \int_0^t a'(t-\tau)\, g(x(\tau))\, d\tau - b'(t) + f'(t). \tag{5.6.8}$$

Because of assumptions (vii), $|b'(t)| \leqslant K$, $\mu \leqslant t < \infty$ if the second alternative holds. If the first alternative holds, we proceed as follows. Since $a''(t) \geqslant 0$, we see that $\gamma''(t) \geqslant 0$, and therefore $-\gamma'(t)$ is nonincreasing, which, together with the last condition in (iv), proves that $|b'(t)| \leqslant K$.

Noting that $a'(t) \in L_1(0, \infty)$, we see, from (5.6.2), (v), (5.6.8), and the hypothesis, that $|x''(t)| \leqslant K$, $\mu < t < \infty$. This, together with (5.6.2)

and the mean value theorem, yields that $|x'(t)| \leqslant K$, $\mu < t < \infty$. Furthermore, from (5.6.1), (iii) and (v) imply $|x'(t)| \leqslant K$, $0 \leqslant t < \infty$.

It is easy to get, after some calculation,

$$V''_+(t) = \Omega_1(t) + \Big\{\tfrac{1}{2}a''(t)\Big(\int_0^t g(x(s))\,ds\Big)^2 + b''(t)\int_0^t g(x(s))\,ds$$
$$+ \tfrac{1}{2}\gamma''(t)\Big\}\exp(-K_1F(t)), \tag{5.6.9}$$

where $V''_+(t)$ is the right-hand derivative of $V'(t)$,

$$\begin{aligned}\Omega_1(t) = &-K_1\,|f(t)|\,V'(t) - K_1V(t)|f(t)|'_+\\ &+\Big\{g'(x(t))\,x'(t)f(t) + g(x(t))[-K_1f(t)\,|f(t)|\\ &+ 2f'(t) - x''(t) - a(0)\,g(x(t))]\\ &- K_1\,|f(t)|\Big(\tfrac{1}{2}a'(t)\Big(\int_0^t g(x(s))\,ds\Big)^2\\ &+ b'(t)\int_0^t g(x(s))\,ds + \tfrac{1}{2}\gamma'(t) - \tfrac{1}{2}\int_0^t a''(t-\tau)\Big(\int_\tau^t g(x(s))\,ds\Big)^2 d\tau\Big)\\ &- \tfrac{1}{2}\int_0^t a'''(t-\tau)\Big(\int_\tau^t g(x(s))\,ds\Big)^2 d\tau\Big\}\exp(-K_1F(t)).\end{aligned}$$

There exists a K such that $\Omega_1(t) \geqslant -K > -\infty$, $\mu \leqslant t < \infty$. This follows from the condition $V(t) \geqslant 0$, $V'(t) \leqslant 0$, the boundedness of $x(t)$, $x'(t)$, $x''(t)$ on $\mu \leqslant t < \infty$, the relation $|f(t)|'_+ = |f'(t)|$, and the hypothesis. Hence, (iv) and (5.6.9) imply that $V''_+(t) \geqslant -K$, $\mu \leqslant t < \infty$. By Lemma 5.5.1 and Remark 5.5.1, we have $V'(t) \to 0$ as $t \to \infty$.

Returning to (5.6.7), we find, as a result of $V'(t) \to 0$ as $t \to \infty$, (i), (iii), (iv), and (v), that

$$\lim_{t\to\infty}\int_0^t a''(t-\tau)\Big(\int_\tau^t g(x(s))\,ds\Big)^2 d\tau = 0. \tag{5.6.10}$$

The arguments used in the proof of Theorem 5.5.1 enable us to conclude from (5.6.10) that $x(t) \to 0$ as $t \to \infty$. From this property, the boundedness of $x''(t)$, and the mean value theorem, we deduce that $x'(t) \to 0$ as $t \to \infty$. The proof is thus complete.

The next theorem concerns the perturbed equation

$$x'(t) = -p(t, x(t)) - \int_0^t a(t-\tau)\,g(x(\tau))\,d\tau + f(t, x(t)). \tag{5.6.11}$$

We define

$$
\begin{aligned}
\hat{g}(x) &= \max_{0 \leqslant \xi \leqslant x} g(\xi), && x \geqslant 0, \\
\hat{g}(x) &= \min_{x \leqslant \xi \leqslant 0} g(\xi), && x \leqslant 0, \\
M(x) &= \max(\hat{g}(x), -\hat{g}(-x)), && x \geqslant 0, \\
m_1(x) &= \min(G(x), G(-x)), && \\
m_2(x) &= \max(G(x), G(-x)), &&
\end{aligned}
\tag{5.6.12}
$$

where

$$G(x) = \int_0^x g(\xi)\, d\xi.$$

Observe that, if $g(x)$ is odd and nondecreasing, then (5.6.12) reduces to $\hat{g}(x) = g(x)$, $M(x) = g(x)$, $m_1(x) = m_2(x) = G(x)$.

THEOREM 5.6.2. Assume that

(i) $a \in C[J, R]$, $(-1)^i a^{(i)}(t) \geqslant 0$ for $0 < t < \infty$, $i = 0, 1, 2$;

(ii) $g \in C[R, R]$, $xg(x) \geqslant 0$ for $|x| \leqslant \rho$, $0 < \rho < \infty$, and $g(x)$ is not identically zero in any neighborhood of the origin;

(iii) $p \in C[J \times R, R]$, $xp(t, x) \geqslant 0$ for $0 \leqslant t < \infty$, $|x| \leqslant \rho$;

(iv) $f \in C[J \times R, R]$, and, for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$, where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, and a $\beta(t) = \beta(t, \epsilon) > 0$, where $\int_0^\infty \beta(t)\, dt < \epsilon$, such that $|f(t, x)| \leqslant \beta(t)$ whenever $0 \leqslant t < \infty$ and $|x| \leqslant \delta$;

(v) for sufficiently small $\epsilon > 0$, $m_1(\delta)/\epsilon M(\delta) > e$.

Then, for any $0 < \eta \leqslant \rho$, there exists a $\delta_0 = \delta_0(\eta)$ such that every solution $x(t)$ of (5.6.11) defined on $0 \leqslant t < \infty$ with $|x(0)| \leqslant x_0$ satisfies

$$|x(t)| < \eta, \qquad t \geqslant 0.$$

Suppose, in addition, that

(vi) $p(t, x)$ is continuously differentiable, and

$$|p(t, x)|, |p_t(t, x)|, |p_x(t, x)| \leqslant K, \qquad 0 \leqslant t < \infty, \qquad |x| \leqslant \rho;$$

(vii) $-a'''(t) \geqslant 0$, $0 < t < \infty$, $a(t) \in L_1(0, \infty)$, $a(t) \not\equiv 0$;

(viii) $xg(x) > 0$, $x \neq 0$, and $g(x)$ is differentiable for $|x| \leqslant \rho$;

(ix) $f(t, x)$ is continuously differentiable, and

$$|f_t(t, x)|, |f_x(t, x)| \leqslant K, \qquad 0 \leqslant t < \infty, \qquad |x| \leqslant \rho;$$

(x) $\beta'_+(t)$, $\beta(t) \leqslant K(\epsilon)$, $0 \leqslant t < \infty$, $0 < \epsilon \leqslant 1$.

Then,

$$\lim_{t \to \infty} x^i(t) = 0 \qquad (i = 0, 1). \tag{5.6.13}$$

Proof. To prove the first assertion of the theorem, let $0 < \eta \leqslant \rho$. Choose $\epsilon = \epsilon(\eta)$, so that $0 < \delta(\epsilon) \leqslant \eta$, and the assumption (v) is satisfied. This means that δ and $\beta(t) = \beta(t, \epsilon)$ are fixed for the remainder of the proof. Now choose $\delta_0 = \delta_0(\eta)$ so that

$$\epsilon M(\delta) + m_2(\delta_0)\, e < m_1(\delta), \tag{5.6.14}$$

which, by the definition of m_1 and m_2, implies that $\delta_0 < \delta$.

Let $x(t)$ be a solution of (5.6.11) defined on J with $|x(0)| \leqslant \delta_0$. Then, by continuity, $|x(t)| < \delta$ for sufficiently small t. Define

$$E(t) = G(x(t)) + \tfrac{1}{2}a(t)\left(\int_0^t g(x(s))\, ds\right)^2 - \tfrac{1}{2}\int_0^t a'(t-\tau)\left(\int_\tau^t g(x(s))\, ds\right)^2 d\tau, \tag{5.6.15}$$

and

$$V(t) = (\epsilon M + E(t)) \exp\left(-\epsilon^{-1}\int_0^t \beta(\tau)\, d\tau\right), \tag{5.6.16}$$

where $M = M(\delta)$. Differentiating (5.6.16), we obtain

$$\begin{aligned} V'(t) = -\frac{1}{\epsilon}\beta(t)\, V(t) + \Big\{&g(x(t))[-p(t, x(t)) + f(t, x(t))] \\ &+ \tfrac{1}{2}a'(t)\left(\int_0^t g(x(s))\, ds\right)^2 - \tfrac{1}{2}\int_0^t a''(t-\tau)\left(\int_\tau^t g(x(s))\, ds\right)^2 d\tau\Big\} \\ &\times \exp\left(-\frac{1}{\epsilon}\int_0^t \beta(\tau)\, d\tau\right). \end{aligned} \tag{5.6.17}$$

Thus, by hypothesis, we have, as long as $|x(t)| \leqslant \delta$, the relation

$$\begin{aligned} V'(t) &\leqslant \{-M\beta(t) + |g(x(t))|\, |f(t, x(t))|\} \exp\left(-\frac{1}{\epsilon}\int_0^t \beta(\tau)\, d\tau\right) \\ &\leqslant 0, \end{aligned}$$

and hence also

$$G(x(t)) \leqslant [\epsilon M + G(x(0))] \exp\left(\frac{1}{\epsilon}\int_0^t \beta(\tau)\,d\tau\right)$$
$$\leqslant (\epsilon M + m_2(\delta_0))\, e < m_1(\delta). \tag{5.6.18}$$

Suppose that there exists a t_1 such that $|x(t_1)| = \delta$. Then, from (5.6.14) and (5.6.18), it follows that

$$m_1(\delta) \leqslant G(x(t)) < m_1(\delta),$$

which is impossible. Hence, no such t_1 exists, and

$$|x(t)| < \delta \leqslant \eta, \qquad t \geqslant 0. \tag{5.6.19}$$

We shall now prove the second part of the assertion. Since $a(t) \in L_1(0, \infty)$, we deduce from (5.6.19), (5.6.11), and the hypothesis the inequality

$$|x'(t)| \leqslant K, \qquad t \geqslant 0.$$

This, together with

$$x''(t) = -p_t(t, x(t)) - p_x(t, x(t))\, x'(t) - a(0)\, g(x(t))$$
$$- \int_0^t a'(t-\tau)\, g(x(\tau))\, d\tau + f_t(t, x(t))$$
$$+ f_x(t, x(t))\, x'(t)$$

and the hypothesis, implies that

$$|x''(t)| \leqslant K, \qquad 0 \leqslant t < \infty.$$

Taking the right derivative of (5.6.17), we obtain the formula

$$V''_+(t) = \tfrac{1}{2}a''(t)\left(\int_0^t g(x(s))\,ds\right)^2 \exp\left(-\frac{1}{\epsilon}\int_0^t \beta(\tau)\,d\tau\right)$$
$$-\frac{1}{\epsilon}\beta(t)\, V'(t) - \frac{1}{\epsilon} V(t)\, \beta'_+(t)$$
$$+ \Big\{ g'(x(t))\, x'(t)[f(t, x(t)) - p(t, x(t))]$$
$$+ g(x(t))\left[\frac{1}{\epsilon}\beta(t)(p(t, x(t)) - f(t, x(t)))\right.$$
$$- 2p_t(t, x(t)) - 2p_x(t, x(t))\, x'(t)$$

$$+ 2f_t(t, x(t)) + 2f_x(t, x(t))\, x'(t) - x''(t)$$

$$- a(0)\, g(x(t))\Big] - \frac{1}{\epsilon}\beta(t)\left(\tfrac{1}{2}a'(t)\left(\int_0^t g(x(s))\, ds\right)^2\right.$$

$$- \tfrac{1}{2}\int_0^t a''(t-\tau)\left(\int_\tau^t g(x(s))\, ds\right)^2 d\tau\Big)$$

$$- \tfrac{1}{2}\int_0^t a'''(t-\tau)\left(\int_\tau^t g(x(s))\, ds\right)^2 d\tau\Big\} \exp\left(-\frac{1}{\epsilon}\int_0^t \beta(\tau)\, d\tau\right).$$

Arguing as in the proof of Theorem 5.6.1, it is easy to deduce that $V''_+(t) \geqslant -K > -\infty$, $t \in J$, and $V'(t) \to 0$ as $t \to \infty$. Hence, it follows from (5.6.17) and the assumptions that (5.6.10) is true. To conclude (5.6.13) from this, we have to repeat the corresponding reasoning as in Theorem 5.6.1. The proof is therefore complete

5.7. Admissibility and asymptotic behavior

In this section, we shall be concerned with an integral equation of the type

$$x(t) = h(t) + \int_0^t K(t, s) f(s, x(s))\, ds. \tag{5.7.1}$$

In order to obtain better results concerning Eq. (5.7.1), we shall need the concept of *admissibility* of a pair of subspaces with respect to an operator.

The underlying space will be the space $C[J, R^n]$, of all continuous functions from J to R^n, with the topology of uniform convergence on every compact interval of J. This topology may be defined by means of seminorms, namely,

$$|x|_n = \sup \|x(t)\|, \qquad 0 \leqslant t \leqslant n, \qquad n = 1, 2, \ldots .$$

It is easy to see that this topology is metrizable and that $C[J, R^n]$ is complete.

Suppose that B, D are Banach spaces of functions from J to R^n such such that $B, D \subset C[J, R^n]$. We shall assume that the topologies of B, D are stronger than the topology of $C[J, R^n]$.

DEFINITION 5.7.1. The pair of spaces (B, D) is said to be admissible with respect to the operator T: $C[J, R^n] \to C[J, R^n]$ iff $TB \subset D$.

LEMMA 5.7.1. Let T be a continuous operator from $C[J, R^n]$ into itself. Suppose that B, D are Banach spaces that are stronger than $C[J, R^n]$ and the pair (B, D) is admissible with respect to T. Then, T is a continuous operator from B to D.

Proof. It is sufficient to show that T is a closed operator from B to D. Then, on the basis of the closed graph theorem, we can conclude that T is a continuous operator from B to D. Let $x_n \xrightarrow{B} x$ and $Tx_n \xrightarrow{D} y$. We must prove that $y = Tx$. From the fact that $x_n \xrightarrow{B} x$, it follows that $x_n \xrightarrow{C[J,R^n]} x$. Consequently, $Tx_n \xrightarrow{C[J,R^n]} Tx$. On the other hand, $Tx_n \xrightarrow{D} y$, and this implies

$$Tx_n \xrightarrow{C[J,R^n]} y.$$

Hence, $y = Tx$, and this means that T is a closed operator, because the graph is closed in $B \times D$. It follows that one can find a constant $k > 0$ such that

$$|\, Tx \,|_D \leqslant k \,|\, x \,|_B , \qquad x \in B.$$

We can now prove an existence theorem for the Volterra integral equation (5.7.1).

THEOREM 5.7.1. Consider Eq. (5.7.1) under the following conditions:

(i) B and D are Banach spaces stronger than $C[J, R^n]$ such that (B, D) is admissible with respect to the operator

$$(Tx)(t) = \int_0^t K(t, s)\, x(s)\, ds, \tag{5.7.2}$$

where $K(t, s)$ is a continuous function for $0 \leqslant s \leqslant t < \infty$.

(ii) $x(t) \to f(t, x(t))$ is a continuous operator on

$$S = [x(t) : x(t) \in D \text{ and } |\, x \,|_D \leqslant \rho],$$

with values in B such that

$$|\, f(t, x(t)) - f(t, y(t))|_B \leqslant \lambda \,|\, x(t) - y(t)|_D , \qquad x, y \in S, \tag{5.7.3}$$

λ being a positive constant.

(iii) $h(t) \in D$.

Then, there exists a unique solution of the Eq. (5.7.1), provided that

$$k\lambda < 1, \qquad |\, h(t)|_D + k \,|\, f(t, 0)|_B \leqslant \rho(1 - \lambda k). \tag{5.7.4}$$

Proof. Consider the following operator on S:

$$(Ux)(t) = h(t) + \int_0^t K(t, x) f(s, x(s))\, ds. \tag{5.7.5}$$

We can write

$$(Ux)(t) - (Uy)(t) = \int_0^t K(t, s)[f(s, x(s)) - f(s, y(s))]\, ds.$$

By Lemma 5.7.1, we get

$$|(Ux)(t) - (Uy)(t)|_D \leqslant \lambda k \mid x(t) - y(t)|_D\,, \tag{5.7.6}$$

taking into account the condition (5.7.3). By the assumption (5.7.4), it follows that U is a contraction operator.

It now suffices to prove that $US \subset S$, in order to conclude the existence and uniqueness of the solution, by means of Banach fixed point theorem.

We have

$$|(Ux)(t)|_D \leqslant \mid h(t)|_D + k \mid f(t, x(t))|_B\,. \tag{5.7.7}$$

But,

$$\begin{aligned} \mid f(t, x(t))|_B &\leqslant |f(t, x(t)) - f(t, 0)|_B + |f(t, 0)|_B \\ &\leqslant \lambda \mid x\mid_D + \mid f(t, 0)|_B\,. \end{aligned}$$

As a result, we obtain

$$|(Ux)(t)|_D \leqslant \mid h(t)|_D + \lambda k\rho + k \mid f(t, 0)|_B < \rho,$$

because of the condition (13.7.4).

The proof is complete.

REMARK 5.7.1. If $f(t, x(t))$ is defined for all $x \in D$, then the second assumption in (5.7.4) is not necessary. Theorem 5.7.1 remains valid if R^n is replaced by a Banach space X. Then $(t, s) \to K(t, s)$ will be an operation into the space of linear bounded operators from X to X.

Another existence theorem can be proved, if we use Schauder's fixed theorem. We shall merely state the theorem, omitting its proof, since it can be proved analogously.

THEOREM 5.7.2. Consider Eq. (5.7.1) under the following conditions:

(i) The same as in Theorem 5.7.1.

(ii) $x(t) \to f(t, x(t))$ is a continuous operator from $\bar{S}$ (the closure of S in $C[J, R^n]$) into B such that

$$|f(t, x(t))|_B \leqslant r,$$

$$\|f(t, x(t))\| \leqslant \lambda(t), \qquad x \in \bar{S}, \qquad t \geqslant 0,$$

where r is a positive constant and $\lambda(t)$ is continuous on R_+.

(iii) The same as in Theorem 5.7.1.

Then, there exists at least one solution $x(t) \in S$ of Eq. (5.7.1), whenever $|h(t)|_D$ and r satisfy the inequality

$$|h(t)|_D + kr \leqslant \rho.$$

In applying Theorems 5.7.1 and 5.7.2 to concrete situations, it is only the admissibility condition (i) that is difficult to be verified. It is therefore important to obtain necessary and sufficient conditions in order that a given pair of spaces is admissible with respect to an integral operator. For this purpose, we shall introduce the space C_g defined as follows.

Let $g(t) > 0$ be continuous on $[0, \infty)$. Then we designate by C_g the space given by

$$C_g = [x(t) \in C[J, R^n] : \|x(t)\| \leqslant Ag(t), t \geqslant 0],$$

where A depends on the function $x(t)$. In the space C_g, we introduce the topology by means of the norm

$$|x|_{C_g} = \sup_{t \in J} \left[\frac{\|x(t)\|}{g(t)}\right].$$

Then it is easy to check that the space C_g is stronger than the space $C[J, R^n]$.

When the spaces B, D are C_g spaces with different g, criteria for their admissibility may be given. The following theorem is to that effect.

THEOREM 5.7.3. Let us consider the pair of spaces (C_g, C_G) and the integral operator T, defined previously. Then the pair (C_g, C_G) is admissible with respect to T iff

$$\int_0^t \|K(t, s)\| g(s)\, ds \leqslant LG(t), \qquad t \geqslant 0, \tag{5.7.8}$$

for some $L > 0$ depending only on g, G, and K.

Proof. Since the sufficiency part of the proof is obvious, we shall only prove that the condition (5.7.8) is necessary.

First of all, let us treat the scalar case, that is, $n = 1$. Suppose that (5.7.8) is not satisfied. Then, we can find a sequence $\{t_m\}$ such that $t_m > 0$ and $t_m \to \infty$ as $m \to \infty$, for which

$$\int_0^{t_m} | K(t_m, s)| g(s) \, ds > mG(t_m), \qquad m \geqslant 1. \tag{5.7.9}$$

Let us define a new function

$$\phi_m(t) = g(t) \operatorname{sign} K(t_m, t), \qquad 0 \leqslant t \leqslant t_m . \tag{5.7.10}$$

This is a measurable function, and

$$| \phi_m(t)| \leqslant g(t).$$

From (5.7.9) and (5.7.10), we have

$$\int_0^{t_m} K(t_m, s) \phi_m(s) \, ds > mG(t_m), \qquad m \geqslant 1. \tag{5.7.11}$$

The theorem of Lucin concerning the structure of measurable functions shows that there exists a continuous function $f_m(t)$, defined on $0 \leqslant t \leqslant t_m$, such that

$$| f_m(t)| \leqslant g(t), \qquad 0 \leqslant t \leqslant t_m ,$$

and

$$\int_0^{t_m} K(t_m, s) f_m(s) \, ds > mG(t_m), \qquad m \geqslant 1. \tag{5.7.12}$$

Without any difficulty, we can extend the function $f_m(t)$ on the whole half-line $t \geqslant 0$ such that it remains continuous and

$$| f_m(t)| \leqslant g(t), \qquad t \in J.$$

Suppose now that $TC_g \subset C_G$. By Lemma 5.7.1 and the fact that $| f_m(t)|_{C_g} \leqslant 1$, $m = 1, 2, \ldots$, it follows that

$$| Tf_m(t)| \leqslant LG(t), \qquad t \in J,$$

for some convenient $L > 0$. This contradicts (5.7.12). Consequently, the condition (5.7.8) is necessary for the admissibility of the pair (C_g, C_G).

In the case when $n > 1$, we shall represent the kernel as a sum

$$K(t, s) = \sum_{i,j=1}^{n} K_{ij}(t, s),$$

where $K_{ij}(t, s)$ is a matrix kernel whose elements are all zero excepting the (i, j)th element. It is easy to see that such a kernel acts as a scalar kernel. As a result, for every $K_{ij}(t, s)$, we can obtain an admissibility of the type (5.7.8). Since

$$\| K(t, s)\| \leqslant \sum_{i,j=1}^{n} \| K_{ij}(t, s)\|,$$

it follows that, for $K(t, s)$ also, we can obtain the condition (5.7.8). The proof is therefore complete.

We shall now derive a particular case of Theorem 5.7.1 as an application.

THEOREM 5.7.4. Consider Eq. (5.7.1) under the following conditions:

(i) $\| K(t, s)\| \leqslant k \exp[-\alpha(t - s)]$, $0 \leqslant s \leqslant t < \infty$, $k, \alpha > 0$.

(ii) $f \in C[J \times S_\rho, R^n]$, $f(t, 0) \equiv 0$, and

$$\|f(t, x) - f(t, y)\| \leqslant \lambda \| x - y \|.$$

(iii) $\| h(t)\| \leqslant h_0 \exp[-\beta t]$, where h_0 and β are positive numbers such that $0 < \beta < \alpha$.

Then, there exists a unique solution of Eq. (5.7.1) satisfying

$$\| x(t)\| \leqslant \rho \exp[-\beta t], \qquad t \in J,$$

whenever h_0 and λ are small enough.

Proof. From condition (i), we obtain

$$\int_0^t \| K(t, s)\| \exp[-\beta s]\, ds \leqslant k(\alpha - \beta)^{-1} \exp[-\beta t], \qquad t \geqslant 0. \tag{5.7.13}$$

This implies that the pair of spaces (C_g, C_g), with $g(t) = \exp[-\beta t]$, is admissible with respect to the operator T, in view of Theorem 5.7.3. Considition (iii) means that $h(t) \in C_g$, and, from (ii), it follows that

$$| f(t, x(t)) - f(t, y(t))|_{C_g} \leqslant \lambda \,| x - y |_{C_g} .$$

Thus, we have verified all the assumptions of Theorem 5.7.1, and hence the proof is complete.

Let us now consider the linear integral equation of the type

$$x(t) = \int_0^t K(t, s)\, x(s)\, ds + f(t), \tag{5.7.14}$$

with continuous kernel for $0 \leqslant s \leqslant t < \infty$ and continuous $f(t)$ on J. The unique solution of (13.7.14) is given by

$$x(t) = \int_0^t R(t, s) f(s)\, ds + f(t), \tag{5.7.15}$$

where $R(t, s)$ is the resolvent kernel of $K(t, s)$, that is,

$$R(t, s) = \sum_{n=1}^{\infty} K_n(t, s), \tag{5.7.16}$$

$$K_1(t, s) = K(t, s),$$

$$K_{n+1}(t, s) = \int_s^t K_n(t, u)\, K(u, s)\, du, \qquad n \geqslant 1. \tag{5.7.17}$$

Corresponding to (5.7.14), we consider the perturbed integral equation

$$x(t) = \int_0^t K(t, s)\, x(s)\, ds + f(t, x). \tag{5.7.18}$$

Definition 5.7.2. Let B, D be Banach spaces such that $B, D \subset C[J, R^n]$ and are stronger than $C[J, R^n]$. Then, the pair (B, D) is said to be admissible for (5.7.14) if the solution $x(t) \in D$ whenever $f(t) \in B$.

We note that this concept of admissibility differs from the one given in Definition 5.7.1.

An argument similar to that utilized in the proof of Lemma 5.7.1 shows that, for every admissible pair (B, D), the mapping $f \to x$ is continuous from B to D. Consequently, for any admissible pair (B, D), we can find a number $k > 0$ such that

$$|\, x\,|_D \leqslant k\, |\, f\,|_B .$$

We are now in a position to prove a result concerning the perturbed equation (5.7.18).

Theorem 5.7.5. Assume that

(i) the pair (B, D) is admissible with respect to (5.7.14), where B and D are Banach spaces stronger than $C[J, R^n]$;

(ii) $x(t) \to f(t, x)$ is a mapping from D to B such that

$$|f(t, x) - f(t, y)|_B \leqslant \lambda \,|\, x - y\,|_D\,, \qquad x, y \in D.$$

Then, there exists a unique solution $x(t) \in D$ of Eq. (5.7.18) provided λ is small enough.

Proof. The proof is very simple. Indeed, consider the mapping $A : u(t) \to x(t)$ from D to D, where $x(t)$ is the solution of the linear equation

$$x(t) = \int_0^t K(t, s)\, x(s)\, ds + f(t, u). \tag{5.7.19}$$

From the admissibility of the pair (B, D), it follows that

$$\begin{aligned} |\, x - y\,|_D &\leqslant k\,|f(t, u) - f(t, v)|_B \\ &\leqslant k\lambda \,|\, u - v\,|_D\,, \end{aligned}$$

where $x = Au, y = Av$. Consequently, for $\lambda < k^{-1}$, the mapping is contractive, and the proof is complete.

Another general theorem that aseerts only the existence of a solution of (5.7.18) may be proved in the same way, making use of Schauder's fixed point theorem.

THEOREM 5.7.6. Let us assume that condition (i) of Theorem 5.7.5 holds. Suppose further that $f(t, x)$ is an operator from D to B such that

(a) $f(t, x)$ is completely continuous;

(b) there exists a number $r > 0$ with the property

$$|f(t, u)|_B \leqslant rk^{-1} \qquad \text{for} \quad |\, u\,|_D \leqslant r.$$

Then, Eq. (5.7.18) has at least one solution $x(t) \in D$ such that $|\, x\,|_D \leqslant r$.

For the proof, it is enough to observe that the mapping $A : u(t) \to x(t)$ defined by (5.7.19) is completely continuous from D to D and carries the ball

$$S = [u : u \in D, |\, u\,|_D \leqslant r] \qquad \text{into itself.}$$

We can derive some concrete results concerning the existence and the behavior of the solutions of perturbed integral equations, as an application of Theorems 5.7.5 and 5.7.6.

Let us suppose that the solution of Eq. (5.7.14) is in C_g for every $f(t) \in C_g$. From (5.7.15), it follows that this situation occurs if and only if

$$\int_0^t R(t, s) f(s)\, ds \in C_g,$$

whenever $f(t) \in C_g$. Hence, on the basis of Theorem 5.7.3, we obtain the necessary and sufficient condition for the pair (C_g, C_g) to be admissible with respect to (5.7.14), namely,

$$\int_0^t |R(t, s)|\, g(s)\, ds \leqslant M g(t), \qquad t \in J, \tag{5.7.20}$$

Suppose, for example, that the operator $f(t, x)$ is given by

$$f(t, x) = \int_0^t K_1(t, s, x(s))\, ds + f(t), \tag{5.7.21}$$

where $f(t) \in C_g$ and $K_1(t, s, x)$ is continuous for $0 \leqslant s \leqslant t < \infty$, $x \in R$ such that $K_1(t, s, 0) = 0$.

In order for the operator defined by (5.7.21) to act from C_g to C_g, it is enough to impose appropriate conditions on $K_1(t, s, x)$. We shall suppose that

$$|K_1(t, s, x) - K_1(t, s, y)| \leqslant K_0(t, s) |x - y|, \tag{5.7.22}$$

where $K_0(t, s)$ is a positive continuous function for $0 \leqslant s \leqslant t < \infty$, satisfying

$$\int_0^t K_0(t, s)\, g(s)\, ds \leqslant M_0 g(t), \qquad t \in J. \tag{5.7.23}$$

Indeed, if $x(t) \in C_g$, then there exists an $A > 0$ such that

$$|x(t)| \leqslant A g(t), \qquad t \in J.$$

From (5.7.21), we obtain

$$\begin{aligned} |f(t, x)| &\leqslant A \int_0^t K_0(t, s)\, g(s)\, ds + |f(t)| \\ &\leqslant A M_0 g(t) + |f(t)|, \end{aligned}$$

and this proves that $f(t, x) \in C_g$ if $f(t) \in C_g$.

The foregoing considerations prove the following result.

THEOREM 5.7.7. Consider the perturbed integral equation

$$x(t) = \int_0^t [K(t, s)\, x(s) + K_1(t, s, x(s))]\, ds + f(t), \tag{5.7.24}$$

subjected to the following conditions:

(i) The pair (C_g, C_g) is admissible for (5.7.14), that is, the resolvent kernel $R(t, s)$ satisfies (5.7.20).

(ii) The function $K_1(t, s, x)$ is continuous for $0 \leqslant s \leqslant t < \infty$, $K_1(t, s, 0) \equiv 0$ and obeys (5.7.22) and (5.7.23).

(iii) $f(t) \in C_g$.

Then, there exists a unique solution of (5.7.24), belonging to C_g, provided M_0 is small enough.

As particular cases of Theorem 5.7.7, we mention the following results concerning the boundedness and the exponential decay of the solutions of the perturbed integral equations. The first case corresponds to the choice $g(t) \equiv 1$, whereas the second one corresponds to

$$g(t) = \exp[-\alpha t],\ \alpha > 0.$$

COROLLARY 5.7.1. Let us suppose that Eq. (5.7.14) has a unique bounded solution for every bounded $f(t)$. Then, the perturbed integral equation

$$x(t) = \int_0^t [K(t, s) + K_0(t, s)]\, x(s)\, ds + f(t) \tag{5.7.25}$$

has a unique bounded solution for every bounded $f(t)$ if

$$\int_0^t |K_0(t, s)|\, ds \leqslant M_0, \qquad t \in J,$$

where M_0 is small enough.

COROLLARY 5.7.2. Assume that (5.7.14) has a solution $x(t)$ verifying

$$|x(t)| \leqslant M \exp[-\alpha t], \qquad \alpha > 0, \qquad t \geqslant 0, \tag{5.7.26}$$

for every $f(t)$ satisfying

$$|f(t)| \leqslant A \exp[-\alpha t], \qquad t \geqslant 0, \tag{5.7.27}$$

where M and A depend on f. Moreover, let $K_0(t, s)$ be a continuous function for $0 \leqslant s \leqslant t < \infty$ with the property that

$$\int_0^t | K_0(t, s)| \exp[\alpha(t - s)]\, ds \leqslant M_1\,, \qquad t \geqslant 0.$$

Then, Eq. (5.7.25) has a unique solution satisfying (5.7.26), whenever $f(t)$ verifies (5.7.27), provided M_1 is small enough.

5.8. Integrodifferential inequalities

Let F be an operator from $C[J, R]$ into $C[J, R]$. We shall consider the integrodifferential equation

$$f(t, x', x, Fx) = 0, \qquad x(0) = x_0\,, \tag{5.8.1}$$

where $f \in C[J \times R^3, R]$. Let us first prove a basic theorem on integrodifferential inequalities.

THEOREM 5.8.1. Let us assume that

(i) $f \in C[J \times R^3, R]$, and $f(t, x, y, z)$ is nondecreasing in x for fixed (t, y, z) and nonincreasing z for fixed (t, x, y);

(ii) the operator F maps $C[J, R]$ into $C[J, R]$, and, for any two functions u_1, $u_2 \in C[J, R]$, the inequality

$$u_1(t) \leqslant u_2(t), \qquad 0 \leqslant t \leqslant t_1\,, \qquad t_1 \in (0, \infty)$$

implies

$$Fu \leqslant Fv \qquad \text{for} \quad t = t_1\,;$$

(iii) $v, w \in C[J, R]$, v, w are continuously differentiable on $(0, \infty)$, and the inequalities

$$f(t, v', v, Fv) \leqslant 0,$$
$$f(t, w', w, Fw) \geqslant 0$$

hold for $t \in (0, \infty)$, one of them being strict.

Then, $v(0) < w(0)$ implies

$$v(t) < w(t), \qquad t \geqslant 0. \tag{5.8.2}$$

Proof. Suppose that the set

$$Z = [t \in J : v(t) \geqslant w(t)]$$

is nonempty. Let $t_1 = \inf Z$. Then $t_1 > 0$, because $v(0) < w(0)$. Furthermore, we have

$$v(t_1) = w(t_1), \tag{5.8.3}$$
$$v(t) \leqslant w(t), \qquad 0 \leqslant t \leqslant t_1,$$

and

$$v'(t_1) \geqslant w'(t_1). \tag{5.8.4}$$

It then follows from assumption (ii) that

$$Fv \leqslant Fw \qquad \text{for} \quad t = t_1. \tag{5.8.5}$$

The monotonicity of the function f now yields

$$\begin{aligned} &f(t_1, v'(t_1), v(t_1), Fv) \\ &\qquad \geqslant f(t_1, w'(t_1), w(t_1), Fw) \end{aligned}$$

because of the relations (5.8.3), (5.8.4), and (5.8.5). This implies a contradiction in view of the strictness of one of the inequalities assumed in (iii). Consequently, the set Z is empty, and (5.8.2) is true. The proof is complete.

REMARK 5.8.1. If the function $f(t, x, y, z)$ is independent of x, then the operator-differential equation (5.8.1) reduces to pure operator equation. Then, for the validity of Theorem 5.8.1, the continuous differentiability of v, w is not necessary.

Remark 5.8.1 may be used to prove the following:

COROLLARY 5.8.1. Let $v, \lambda \in C[J, R_+]$, and suppose that

$$v(t) \leqslant v_0 + \int_0^t \lambda(s)\, ds,$$

where $v_0 > 0$ is a constant. Then,

$$v(t) \leqslant v_0 \exp\left[\int_0^t \lambda(s)\, ds\right], \qquad t \geqslant 0.$$

Proof. To apply Theorem 5.8.1, we set

$$f(t, x, y, z) = y - z - v_0$$

and

$$Fu = \int_0^t \lambda(s)\, u(s)\, ds.$$

Consider the function $w(t) = (v_0 + \epsilon) \exp[\int_0^t \lambda(s)\, ds]$ for arbitrary small $\epsilon > 0$. Then, it is easy to check that

$$f(t, v, Fv) \leqslant 0,$$

$$f(t, w, Fw) > 0,$$

and

$$v_0 < w(0).$$

Since the assumptions of Theorem 5.8.1 hold, we have

$$v(t) < w(t), \qquad t \geqslant 0.$$

As this inequality is true for all $\epsilon > 0$, we deduce, letting $\epsilon \to 0$, the desired result.

It is not difficult to see that Theorem 5.8.1 includes integrodifferential equations of the form

$$x'(t) = f(t, x(t)) + \int_0^t K(t, s, x(s))\, ds,$$

where the kernel K is monotone nondecreasing.

DEFINITION 5.8.1. Let $v \in C[J, R]$, and $v'(t)$ exist and be continuous for $t \in (0, \infty)$. If v satisfies the inequality

$$f(t, v', v, Fv) > 0, \qquad t \in (0, \infty),$$

then $v(t)$ is said to be an *over function* with respect to the integrodifferential equation (5.8.1). On the other hand, if v satisfies

$$f(t, v', v, Fv) < 0, \qquad t \in (0, \infty),$$

then $v(t)$ is said to be an *under function*.

As a consequence of Theorem 5.8.1, we have the following result.

THEOREM 5.8.2. Let $u(t)$, $w(t)$ be under and over functions with respect to Eq. (5.8.1) and $v(t)$ be a solution of (5.8.1) existing on $[0, \infty)$. Then

$$u(0) < v(0) < w(0)$$

implies

$$u(t) < v(t) < w(t), \qquad t \geqslant 0.$$

DEFINITION 5.8.2. Let $v \in C[J, R]$, and $v'(t)$ exist and be continuous for $0 < t < \infty$. Then $v(t)$ is said to be a *δ-approximate solution* of the integrodifferential equation (5.8.1), if $v(t)$ satisfies the inequality

$$|f(t, v'(t), v(t), Fv)| \leqslant \delta(t), \qquad t \in (0, \infty),$$

where $\delta \in C[J, R_+]$.

A result that gives an error estimation of the δ-approximate solution is the following:

THEOREM 5.8.3. Let $v(t)$ be a δ-approximate solution of (5.8.1). Suppose further that

$$\begin{aligned} &f(t, x_1, y_1, Fy_1) - f(t, x_2, y_2, Fy_2) \\ &\qquad \geqslant g(t, x_1 - x_2, y_1 - y_2, G(y_1 - y_2)), \qquad x_1 \geqslant x_2, \qquad y_1 \geqslant y_2, \end{aligned}$$

where $g \in C[J \times R^3, R]$, and G is an operator that maps $C[J, R]$ into $C[J, R]$. Assume that the function $g(t, x, y, z)$ is nondecreasing in x for fixed (t, y, z) and nonincreasing z for fixed (t, x, y), and, for any two functions u, $v \in C[J, R]$, the inequality

$$u(t) \leqslant v(t), \qquad 0 \leqslant t \leqslant t_1, \qquad t_1 \in (0, \infty),$$

implies

$$Gu \leqslant Gv \qquad \text{for} \quad t = t_1 .$$

Then, if $u(t)$ is any solution of (5.8.1) such that $u(0) = x_0$ and $| v(0) - x_0 | \leqslant \rho_0$, we have

$$| v(t) - u(t)| < \rho(t), \qquad t \geqslant 0,$$

where $\rho(t) > 0$ is continuously differentiable for $0 < t < \infty$ and satisfies

$$g(t, \rho', \rho, G\rho) > \delta(t), \qquad t \in (0, \infty).$$

Proof. We shall first show that $v(t) - u(t) < \rho(t)$, $t \geqslant 0$. Setting $z(t) = v(t) - u(t)$ and proceeding as in the proof of Theorem 5.8.1, we arrive at a $t_1 > 0$ with the properties

$$z(t_1) = \rho(t_1),$$

$$z'(t_1) \geqslant \rho'(t_1),$$

and

$$Gz \leqslant G\rho, \qquad t = t_1 .$$

Since $\rho(t_1) > 0$, $\rho'(t_1) \geqslant 0$, we have $v(t_1) \geqslant u(t_1)$, $v'(t_1) \geqslant u'(t_1)$, and, consequently,

$$\begin{aligned}\delta(t_1) &\geqslant f(t_1, v'(t_1), v(t_1), Fv) - f(t_1, u'(t_1), u(t_1), Fu)\\ &\geqslant g(t_1, z'(t_1), z(t_1), Gz).\end{aligned}$$

Now, using the monotonicity properties of g, it follows that

$$\begin{aligned}&g(t_1, z'(t_1), z(t_1), Gz)\\ &\qquad\leqslant g(t_1, \rho'(t_1), \rho(t_1), G\rho)\\ &\qquad< \delta(t_1),\end{aligned}$$

which implies $\delta(t_1) < \delta(t_1)$. This absurdity proves

$$v(t) - u(t) < \rho(t), \qquad t \geqslant 0.$$

A similar argument shows that $u(t) - v(t) < \rho(t)$, $t \geqslant 0$. The theorem is therefore proved.

5.9. Notes

See Walter [3] for the type of results in Sect. 5.1 (see Jones [1]). Theorems 5.2.1 and 5.2.3 are due to Nohel [1]. Theorem 5.2.2 is new. For Theorem 5.3.1, see Nohel [1]. The proof of Theorem 5.3.2 is adopted from Olech [9], whereas Corollary 5.3.1 is due to Olech [9]. See also Cafiero [1].

For the results of the type given in Sect. 5.4, see Walter [3]. Sections 5.5 and 5.6 consist of the work of Levin and Nohel [2, 3]. See also Friedman [1, 2], Halanay [3], Levin [2, 3], Miller [5], and Padmavally [1].

The results of Sect. 5.7 are due to Corduneanu [18, 21]. Section 5.8 contains results adopted from Nickel [1]. See also Azbelev and Tzaliuk [1], Barbu [1], Baumann [1], Beneš [1, 2], Cameron and Shapiro [1], Corduneanu [19], Erdelyi [1], Goldenhershel [1], Iwasaki and Sato [1], Krasnoselskii [1], Krein [2], Levin and Nohel [4], Mann and Roberts [1], Miller [6, 7], Mitryakov [1], Nohel [2–5], Petrovanu [1], Ramamohana Rao [2], Sato [1, 4], Volterra [1–3], and Willett [1].

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